CPSC 768:
Scalable and Private Graph Algorithms

Lecture 7: Streaming Maximum Matching

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Announcements

- Check the latest announcement on Canvas:
    - Scheduling lectures
    - Link for joining CPSC 768 Slack

Last Time: Maximum Matching in Bounded Arboricity Graphs

- **Problem**: Given an insertion-only arbitrary-order stream of edges, find an approximate size of the maximum matching in the graph using small space

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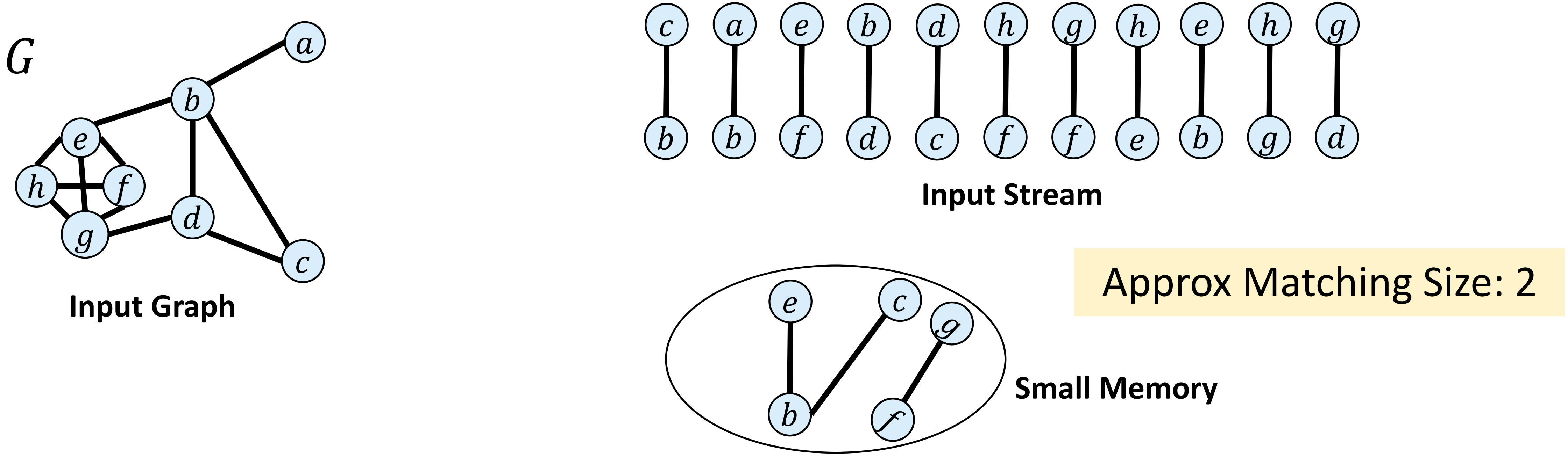
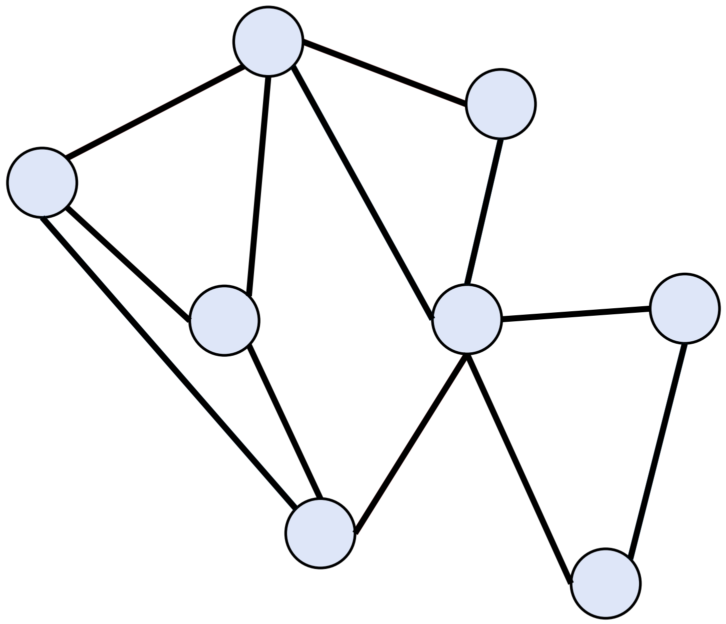
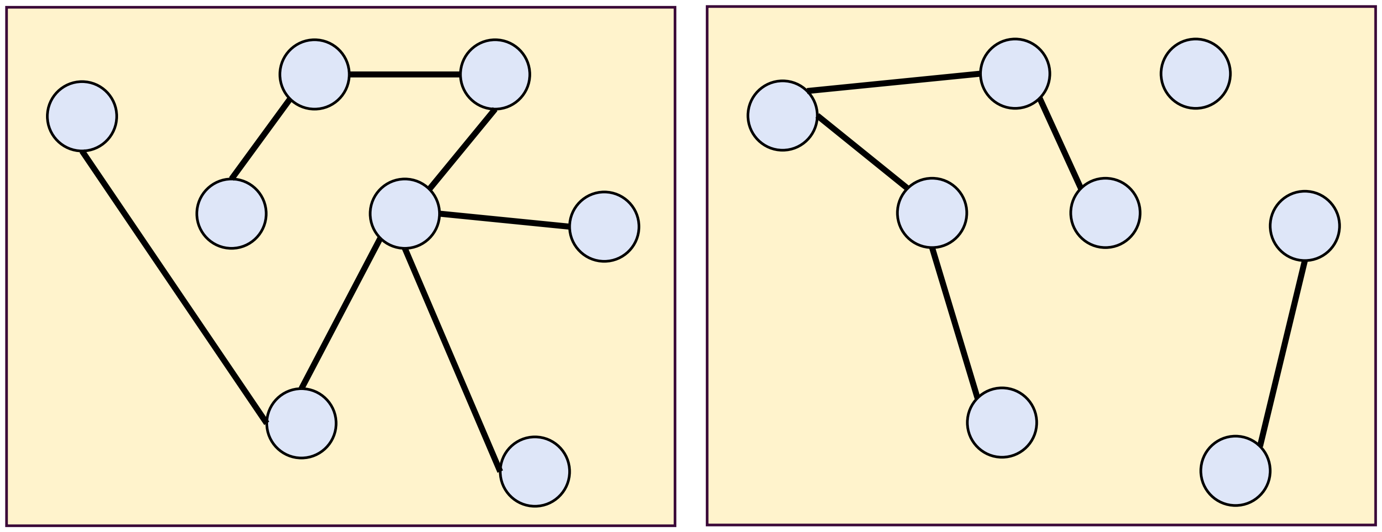


Arboricity of the Graph

- **Arboricity** of the graph
  - Minimum number of forests to decompose the graph





Properties of Arboricity

- Related to the **density** of the graph

By Nash-Williams Theorem:

$$\boldsymbol{\alpha} = \max_{\boldsymbol{S}} \left\{ \left\lceil \frac{\boldsymbol{m_S}}{\boldsymbol{n_S - 1}} \right\rceil \right\}$$

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- Every subgraph $S \subseteq G$ **has arboricity** $\leq \alpha$

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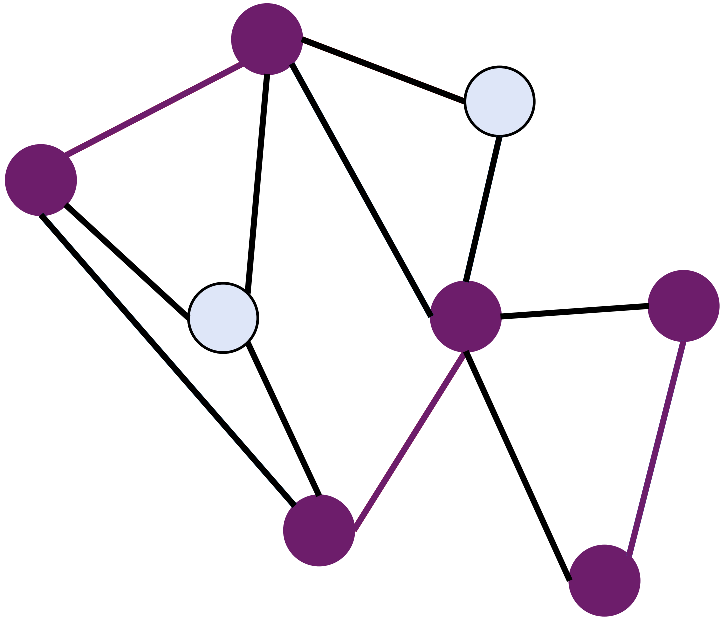
- Every subgraph $S \subseteq G$ **has arboricity** $\leq \alpha$
- Subgraph S has at most $\alpha \cdot V(S)$ **edges**

Maximum Matching

- A **matching** in a graph is a set of edges where no two edges share an endpoint



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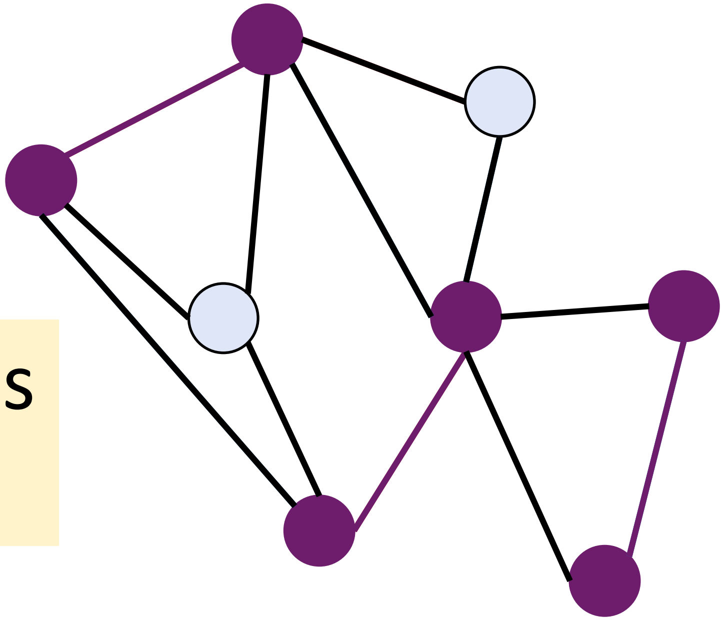
Maximum Matching has **Maximum Cardinality**



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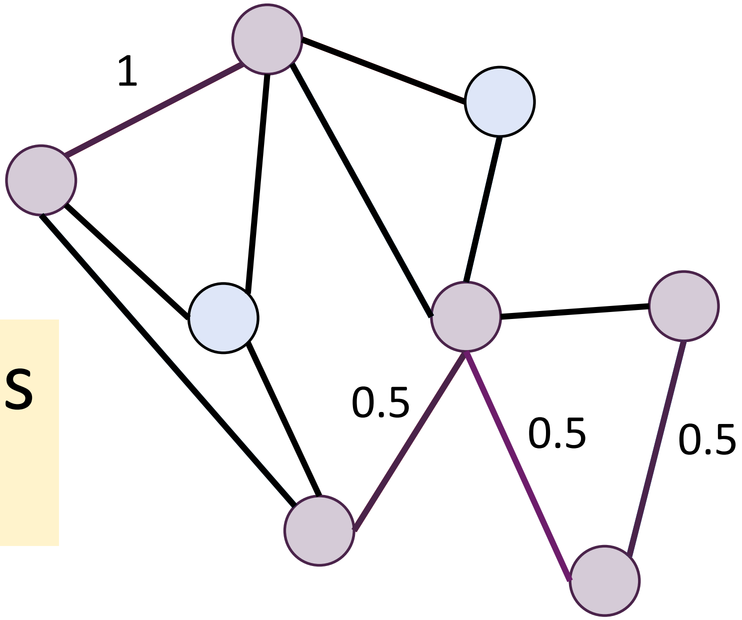

Fractional Matching:

$$\forall v \in V: \sum_{e \ni v} f(e) \leq 1$$

Maximum Matching has **Maximum Cardinality**

Strategy for Streaming Algorithms

1. Figure out quantity to approximate and gives approximation of the quantity we want to approximate
2. Approximate the quantity via sampling and prove concentration bounds

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Strategy for Streaming Algorithms

1. **Approximate $|E_\alpha|$** to approximate **$M(G)$ the maximum matching size**
   - E_α is set of edges $\{u, v\}$ where **u and v both incident to at most α edges** that show up **later in the stream**

Lemma 1: $M(G) \leq |E_\alpha| \leq (\alpha + 2) \cdot M(G)$

Strategy for Streaming Algorithms

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Last time: Proved $|E_\alpha| \leq (\alpha + 2) \cdot M(G)$ via defining fractional matching $Y_e = \frac{1}{\alpha+1}$ if $e \in E_\alpha$ and 0 otherwise

Edmond's Matching Polytope Corollary:

Let $\{Y_e\}_{e \in E}$ be a fractional matching where the **maximum weight** on any edge is $\boldsymbol{\eta}$. Then, $\boldsymbol{\sum_{e \in E} Y_e \leq (1 + \eta) \cdot M(G)}$.

Strategy for Streaming Algorithms

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Last time: Proved $|E_\alpha| \leq (\alpha + 2) \cdot M(G)$ via defining fractional matching $Y_e = \frac{1}{\alpha+1}$ if $e \in E_\alpha$ and 0 otherwise.

Thus, $\frac{1}{\alpha+1} \cdot |E_\alpha| \leq \left(1 + \frac{1}{\alpha+1}\right) \cdot M(G) = \frac{\alpha+2}{\alpha+1} \cdot M(G)$ and so

$$|E_\alpha| \leq (\alpha + 2) \cdot M(G)$$

Strategy for Streaming Algorithms

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- Defined **good** edge $\{u, v\} \in B_u \cap B_v$

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- Defined **good** edge $\{u, v\} \in B_u \cap B_v$
- Defined **wasted** edge $\{a, b\} \in B_a \oplus B_b$

E_α is **exactly** set of good edges

Strategy for Streaming Algorithms

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- $\boldsymbol{w}$:= number of good edges with **no** endpoints in H

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- w := number of good edges with **no** endpoints in H
- x := number of good edges with **one** endpoint in H

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- First, figure out $x + y$ by stating some facts

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  2. Good and wasted edges:

     $z + y \leq \alpha \cdot |H|$

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All edges in H:
at most $\alpha \cdot |H|$ of them

- First, figure out $x + y$ by stating some facts
  1. Number of edges in the **$\boldsymbol{B_u}$ of every $\boldsymbol{u \in H}$ incident to good edge**
     - $(\alpha + 1)|H| = x + 2y + z$
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- w := **no** endpoints in H
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- Hence,
  - $x + 2y + z = (\alpha + 1)H$

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Define $\boldsymbol{E_L}$ be set of edges with **no endpoints in $\boldsymbol{H}$**

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- **Every edge in E_L is good**

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  - $\boldsymbol{x} + \boldsymbol{y} \geq |\boldsymbol{H}|$
- **Every edge in $\boldsymbol{E_L}$ is good**
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- Therefore,

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- Therefore,
  $\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{w} \geq |\boldsymbol{H}| + |\boldsymbol{E_L}|$

Strategy for Streaming Algorithms

$$|E_\alpha| = w + x + y \geq |\boldsymbol{H}| + |\boldsymbol{E_L}|$$

- **Relate back to $\boldsymbol{M(G)}$**

Define $\boldsymbol{E_L}$ be set of edges with **no endpoints in $\boldsymbol{H}$**

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Strategy for Streaming Algorithms

$$|E_\alpha| = w + x + y \geq |\boldsymbol{H}| + |\boldsymbol{E_L}|$$

- **Relate back to $M(G)$**
  - What is the size of $M(G)$ in relation to $|H|$ and $|E_L|$?

Define $\boldsymbol{E_L}$ be set of edges with **no endpoints in $\boldsymbol{H}$**

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Strategy for Streaming Algorithms

$$|E_\alpha| = w + x + y \geq |\boldsymbol{H}| + |\boldsymbol{E_L}|$$

- **Relate back to $\boldsymbol{M(G)}$**
  - What is the size of $M(G)$ in relation to $|H|$ and $|E_L|$?
  - Every edge in E_L could be in matching

Define $\boldsymbol{E_L}$ be set of edges with **no endpoints in $\boldsymbol{H}$**

- Hence,
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- **Relate back to $\boldsymbol{M(G)}$**
  - What is the size of $M(G)$ in relation to $|H|$ and $|E_L|$?
  - Every edge in E_L could be in matching
  - Remaining edges incident to H
    - At most one edge incident to each $u \in H$

- Hence,
  - $x + 2y + z = (\alpha + 1)H$
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  - $\boldsymbol{x} + \boldsymbol{y} \geq |\boldsymbol{H}|$
- **Every edge in $\boldsymbol{E_L}$ is good**
- $\boldsymbol{w} = |\boldsymbol{E_L}|$
- Therefore,

  $\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{w} \geq |\boldsymbol{H}| + |\boldsymbol{E_L}|$

Strategy for Streaming Algorithms

$$|E_\alpha| = w + x + y \geq |\boldsymbol{H}| + |\boldsymbol{E_L}| \geq \boldsymbol{M}(\boldsymbol{G})$$

- **Relate back to $\boldsymbol{M}(\boldsymbol{G})$**
  - What is the size of $M(G)$ in relation to $|H|$ and $|E_L|$?
  - Every edge in E_L could be in matching
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- Hence,
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- **Every edge in $\boldsymbol{E_L}$ is good**
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- Therefore,
  $\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{w} \geq |\boldsymbol{H}| + |\boldsymbol{E_L}|$

Strategy for Streaming Algorithms

- **Most of our work is proving:**

Lemma 1: $M(G) \leq |E_\alpha| \leq (\alpha + 2) \cdot M(G)$

Strategy for Streaming Algorithms

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Approximating $|E_\alpha|$

- Let $\boldsymbol{G_t}$ be the graph defined by the prefix of the stream consisting of first $\boldsymbol{t}$ **edges**

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  - Let $\boldsymbol{E_\alpha^t}$ be the set of good edges in this prefix

Approximating $|E_\alpha|$

- Let $\boldsymbol{G_t}$ be the graph defined by the prefix of the stream consisting of first $\boldsymbol{t}$ **edges**
  - Let $\boldsymbol{E_\alpha^t}$ be the set of good edges in this prefix
  - Let $\boldsymbol{E^*} = \max_{\boldsymbol{t}} (|\boldsymbol{E_\alpha^t}|)$

Then, $\boldsymbol{M(G) \leq E^* \leq (\alpha + 2) \cdot M(G)}$
since $E^* \geq |E_\alpha|$ and $M(G_t) \leq M(G)$

Approximating $|E_\alpha|$

- Let $\boldsymbol{G_t}$ be the graph defined by the prefix of the stream consisting of first $\boldsymbol{t}$ **edges**
  - Let $\boldsymbol{E_\alpha^t}$ be the set of good edg
  - Let $\boldsymbol{E^* = \max_t (|E_\alpha^t|)}$

Question: does $|E_\alpha^t|$ ever drop as t increases?

Then, $\boldsymbol{M(G) \leq E^* \leq (\alpha + 2) \cdot M(G)}$ since $E^* \geq |E_\alpha|$ and $M(G_t) \leq M(G)$

Approximating E^*

Theorem: Can approximate E^* to $(1+\varepsilon)$-approximation in $O\left(\frac{\log(n)}{\varepsilon^2}\right)$ space whp.

Approximating E^*

- Intuition: sample edges from E_α^t to obtain accurate approximation of $|E_\alpha^t|$

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Approximating E^* Algorithm

- Intuition: sample edges from E_α^t to obtain accurate approximation of $|E_\alpha^t|$
- For each sampled edge $e = \{u, v\}$, **store c_e^u and c_e^v** for degrees of u and v in the rest of the stream

Approximating E^* Algorithm

- Intuition: sample edges from E_α^t to obtain accurate approximation of $|E_\alpha^t|$
- For each sampled edge $e = \{u, v\}$, **store $\boldsymbol{c_e^u}$ and $\boldsymbol{c_e^v}$** for degrees of u and v in the rest of the stream
  - If either $\boldsymbol{c_e^u}$ or $\boldsymbol{c_e^v}$ exceeds α delete $\{u, v\}$

Approximating E^* Algorithm

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1. Initialize $S \leftarrow \emptyset$, $p = 1$, estimate = 0
2. For each $\{u, v\}$ in stream:
   a) With probability p add $S \leftarrow S \cup \{u, v\}$, initialize counters

Add new sampled edges

Approximating E^* Algorithm

1. Initialize $S \leftarrow \emptyset$, $p = 1$, estimate = 0
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    a) With probability p add $S \leftarrow S \cup \{u, v\}$, initialize counters
    b) For each edge $e' \in S$, if e' shares endpoint w with e:

Approximating E^* Algorithm

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   a) With probability p add $S \leftarrow S \cup \{u, v\}$, initialize counters
   b) For each edge $e' \in S$, if e' shares endpoint w with e:
      i. Increment $c_{e'}^{w}$

Check the counters of previously sampled edges

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2. For each $\{u, v\}$ in stream:
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   b) For each edge $e' \in S$, if e' shares endpoint w with e:
      i. Increment $c_{e'}^{w}$
      ii. If $c_{e'}^{w} > \alpha$, remove e' and corresponding counters from S

Remove edge if it is no longer in E_{α}^{t}

Approximating E^* Algorithm

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   c) If $|S| > 80\, \varepsilon^{-2} \log n$:

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      i. Increment $c_{e'}^{w}$
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   c) If $|S| > 80\,\varepsilon^{-2} \log n$:
      i. Set $p \leftarrow \frac{p}{2}$

If you used too much space, reduce sampling rate

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   a) With probability p add $S \leftarrow S \cup \{u, v\}$, initialize counters
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      i. Increment $c_{e'}^{w}$
      ii. If $c_{e'}^{w} > \alpha$, remove e' and corresponding counters from S
   c) If $|S| > 80\, \varepsilon^{-2} \log n$:
      i. Set $p \leftarrow \frac{p}{2}$
      ii. Remove each edge in S with probability ½

Resample previous samples

Approximating E^* Algorithm

1. Initialize $S \leftarrow \emptyset$, $p = 1$, estimate = 0
2. For each $\{u, v\}$ in stream:
   a) With probability p add $S \leftarrow S \cup \{u, v\}$, initialize counters
   b) For each edge $e' \in S$, if e' shares endpoint w with e:
      i. Increment $c_{e'}^{w}$
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   c) If $|S| > 80\,\varepsilon^{-2} \log n$:
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   d) Estimate $\leftarrow \max($estimate, $|S|/p)$

Update estimate of E^*

Approximating E^* Algorithm

Theorem: Can approximate E^* to $(1+\varepsilon)$-approximation in $O\left(\frac{\log(n)}{\varepsilon^2}\right)$ space whp.

- Proof: Let $\boldsymbol{\tau} = \frac{\mathbf{40 \log n}}{\boldsymbol{\varepsilon^2}}$ and level i (starting with $i = 2$) be
  $$\mathbf{2^{i-1}} \cdot \boldsymbol{\tau} \leq |\boldsymbol{E_\alpha^t}| < \mathbf{2^i} \cdot \boldsymbol{\tau}$$
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Need to prove that p_i matches the level whp

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- Take the union bound over $t \leq n^2$, then with probability at least $1 - \frac{1}{\text{poly}(n)}$:

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Lemma 1: $M(G) \le E^* \le (\alpha + 2) \cdot M(G)$

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Approximating E^* Algorithm

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  pol...
  - ...
    all i

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Theorem: Can approximate $\boldsymbol{M(G)}$ to $\boldsymbol{(2+\alpha)(1+\varepsilon)}$**-approximation** in $\boldsymbol{O\left(\frac{\log(n)\cdot\log(\alpha)}{\varepsilon^2}\right)}$ **space whp.**

On Wednesday

- Streaming Bipartite Matching using **Auction Algorithms**
  - Another more intuitive way to solve maximum matching in bipartite graphs than Hungarian algorithm or maxflow!