CPSC 768: Scalable and Private Graph Algorithms

Lecture 5: Approximate Average Degree in the Sublinear Model; Arboricity and Orientation

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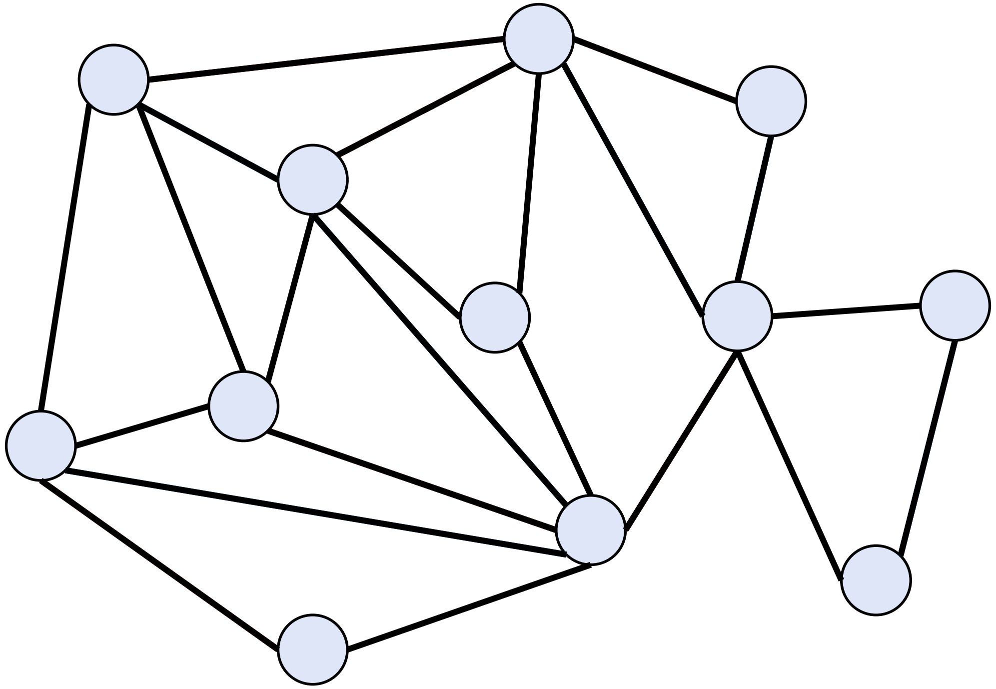
Outdegree Orientation

- $(1 + \varepsilon)$-approx. for average degree + useful graph property!



Orient all edges from low to high degree, what's the max out-degree that you see?

Give an Ordering of the Vertices to Minimize Outdegree

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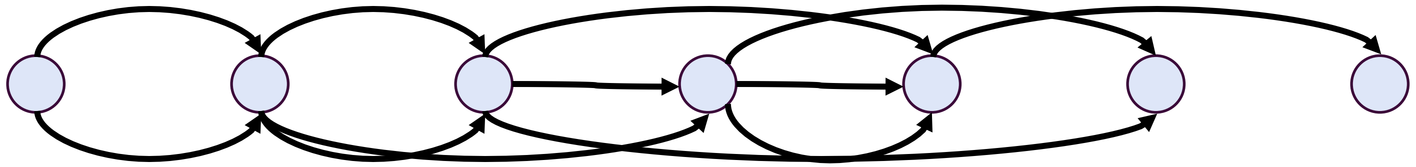

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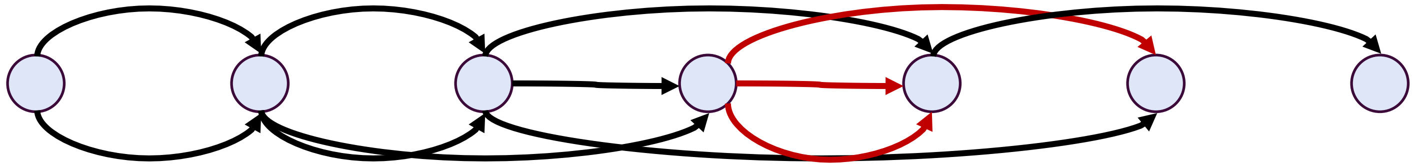
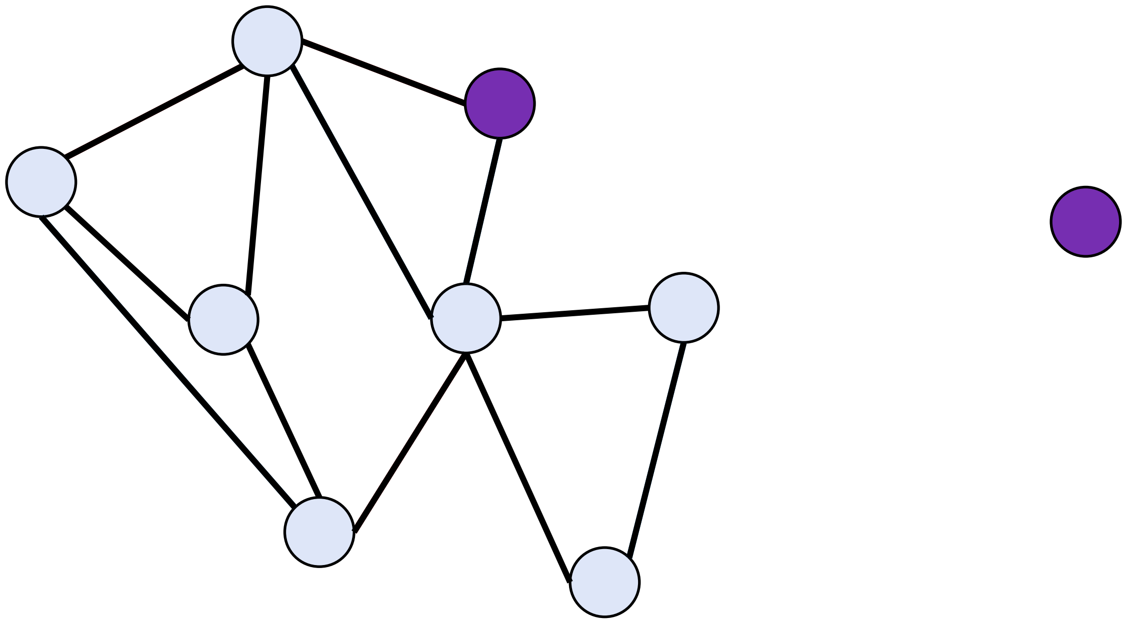


- Produces an ordering that minimizes the *maximum* outdegree

Ideas for an algorithm to achieve this?

Linear Time $O(n + m)$ Solution

- Repeated peel vertex with minimum remaining induced degree and put in order



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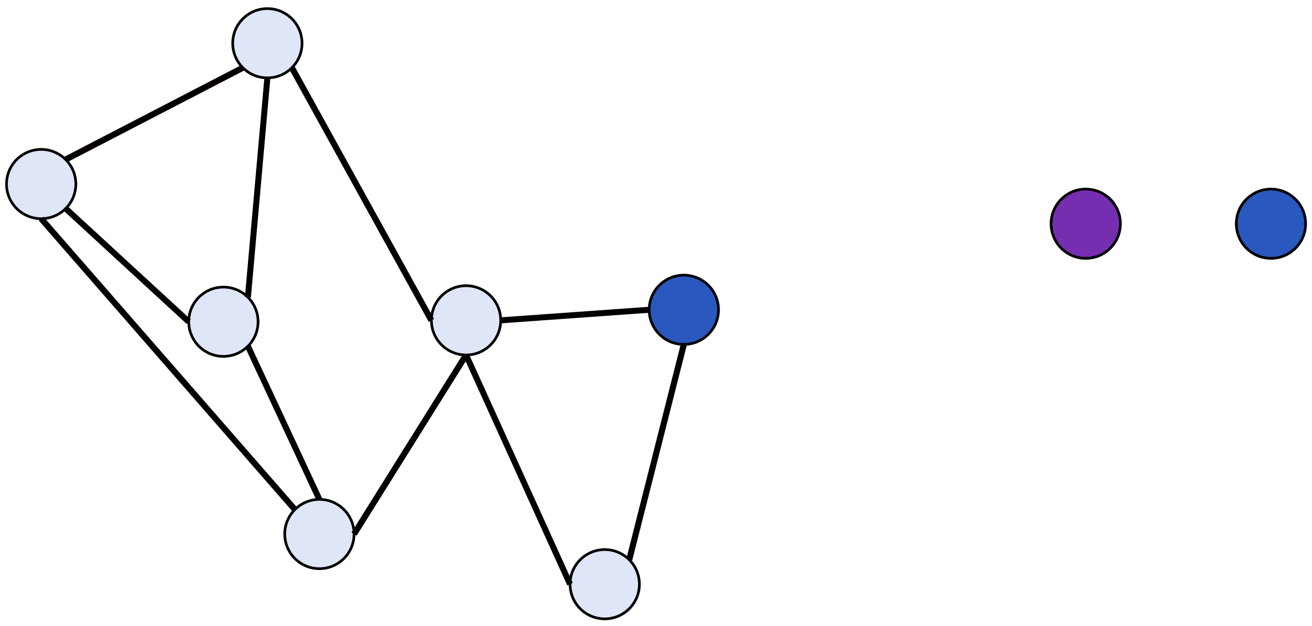

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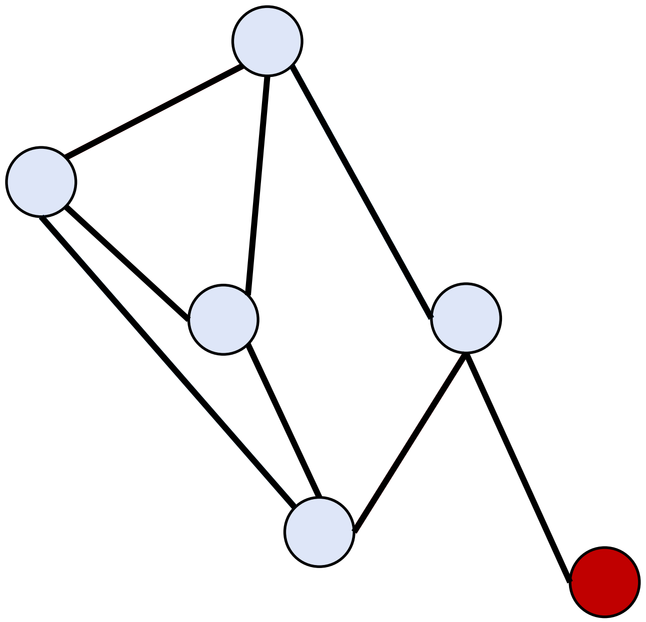

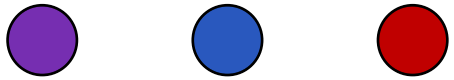

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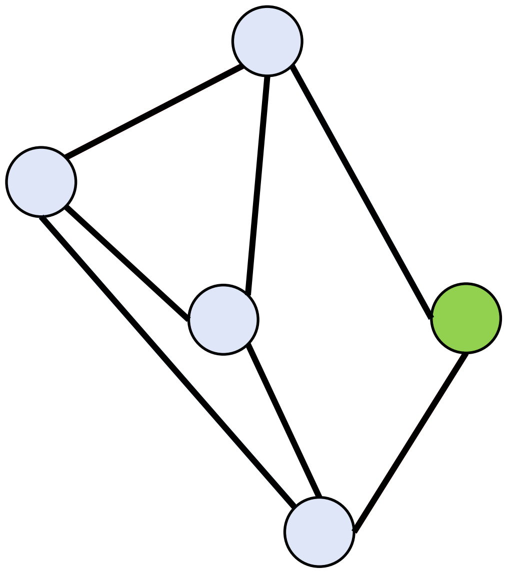

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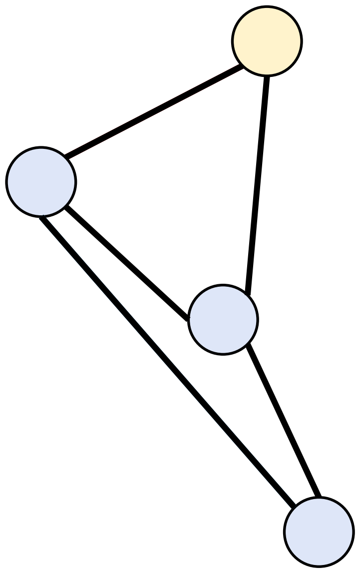

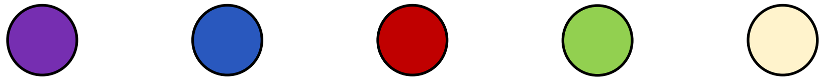

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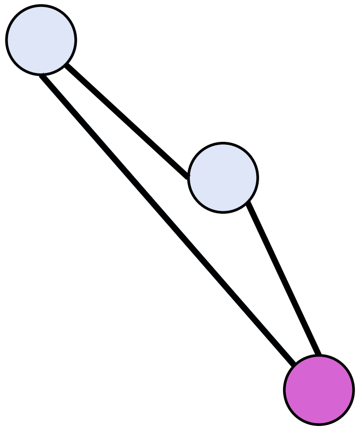

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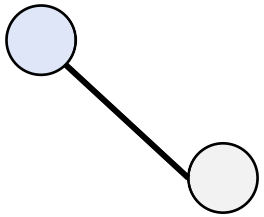

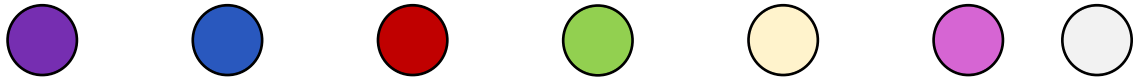

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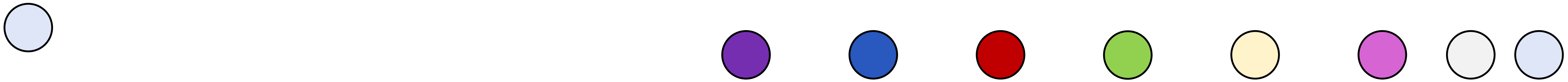

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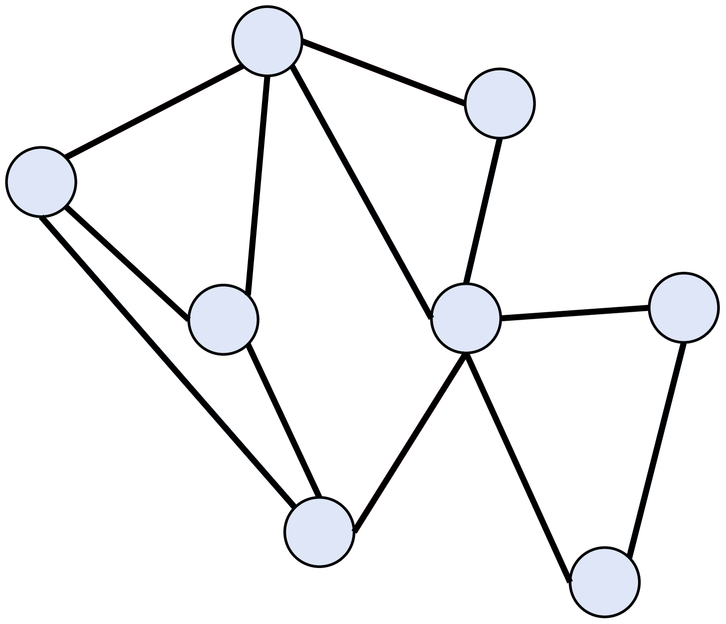

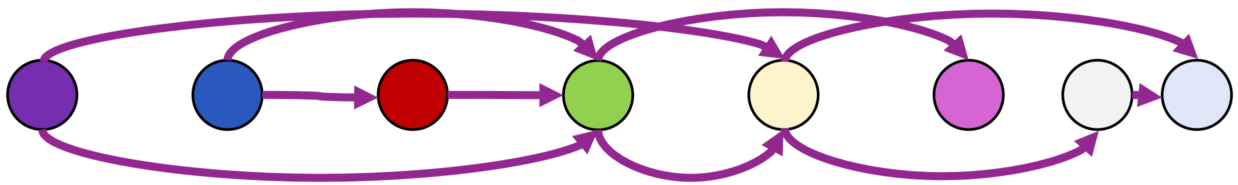

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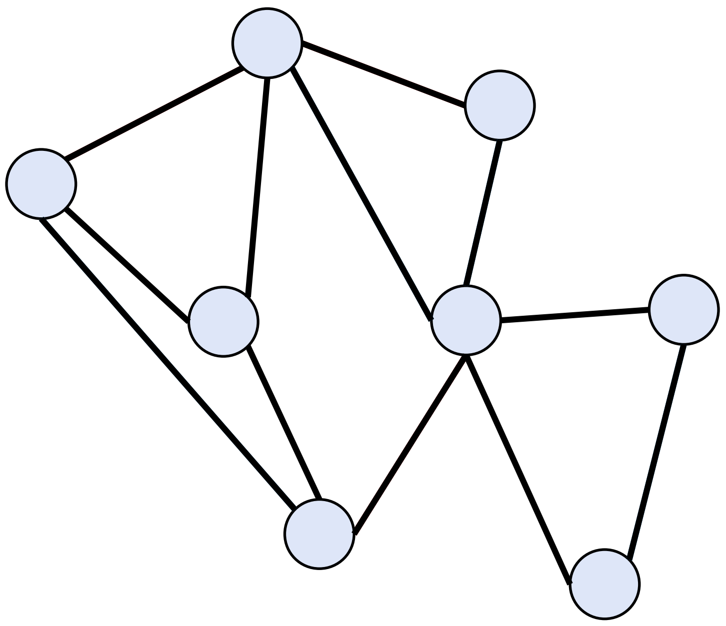
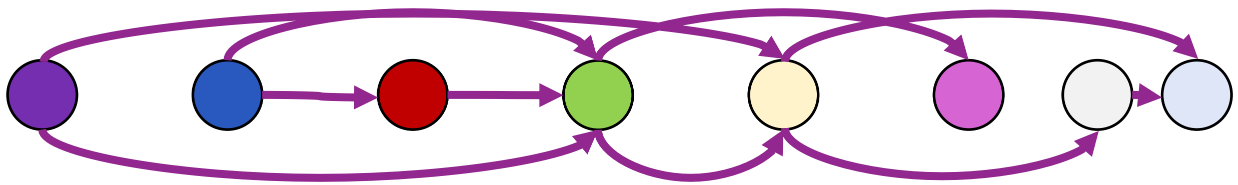


Outdegree: 2



Can make linear time by using n buckets in $O(n)$ space

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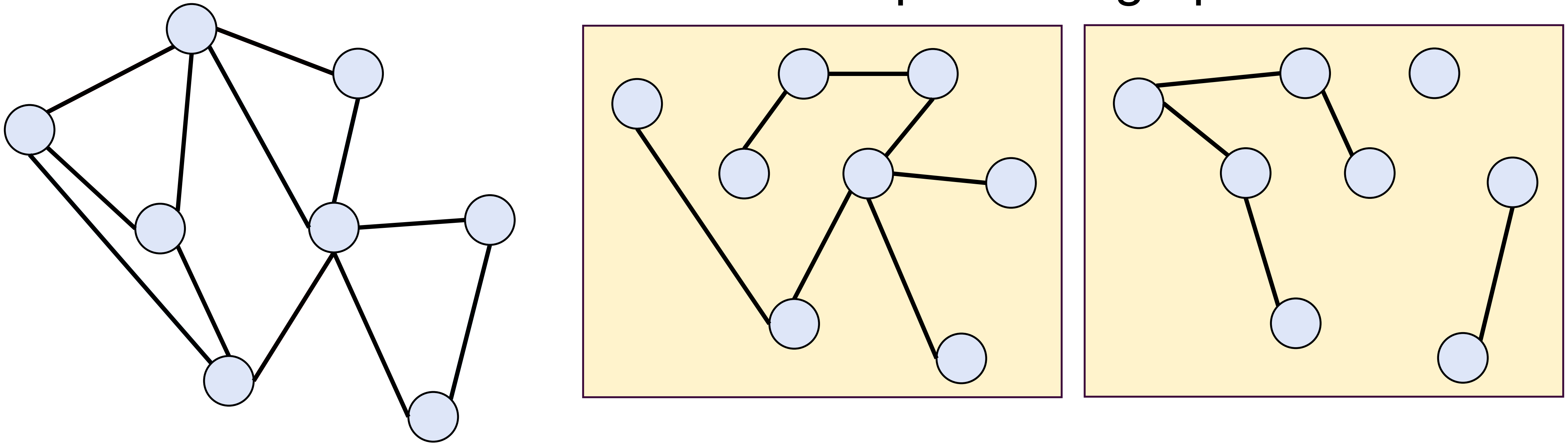

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$$\frac{\boldsymbol{d}}{\boldsymbol{2}} \leq \boldsymbol{\alpha} \leq \boldsymbol{d}$$

By Nash-Williams Theorem:

$$\boldsymbol{\alpha} = \max_{\boldsymbol{s}} \left\{ \left\lceil \frac{\boldsymbol{m_s}}{\boldsymbol{n_s} - \boldsymbol{1}} \right\rceil \right\}$$

Degeneracy is Very Small in Real-Life

Graph	Num. Vertices	Num. Edges	*d*
dblp	425,957	2,099,732	113
brain-network	784,262	267,844,669	1200
wikipedia	1,140,149	2,787,967	124
youtube	1,138,499	5,980,886	51
stackoverflow	2,601,977	28,183,518	198
livejournal	4,847,571	85,702,474	372
orkut	3,072,627	234,370,166	253
usa-central	14,081,816	16,933,413	3
usa-road	23,072,627	28,854,312	3
twitter	41,652,231	1,202,513,046	2488
friendster	65,608,366	1,806,067,135	304

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Simple Algorithm (Chiba-Nishizeki '85)

1. For every edge, count number of triangles **incident to lower degree endpoint**
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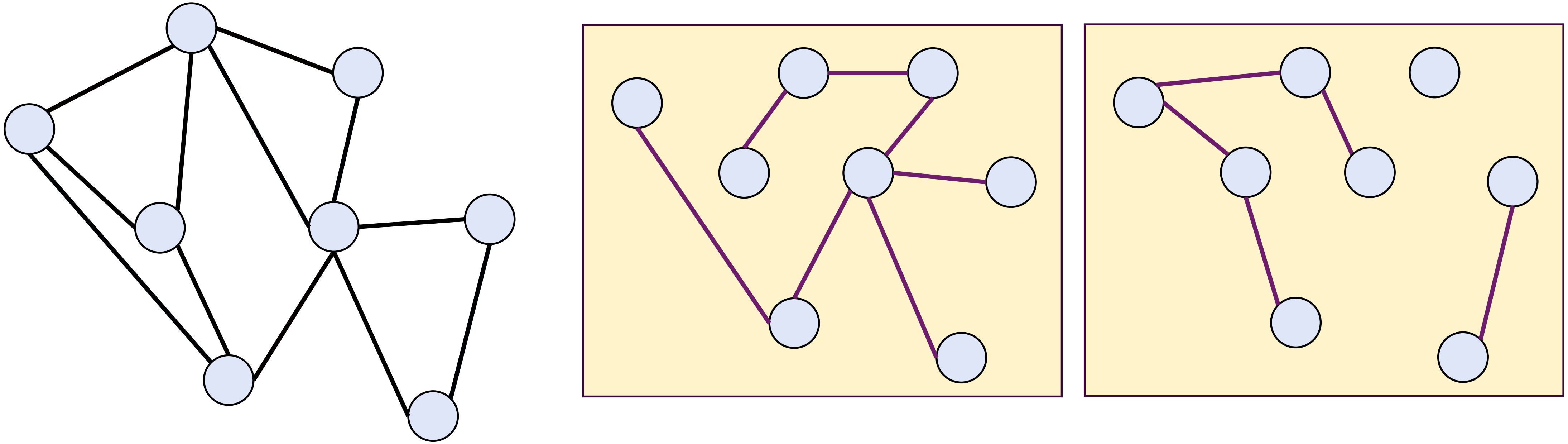

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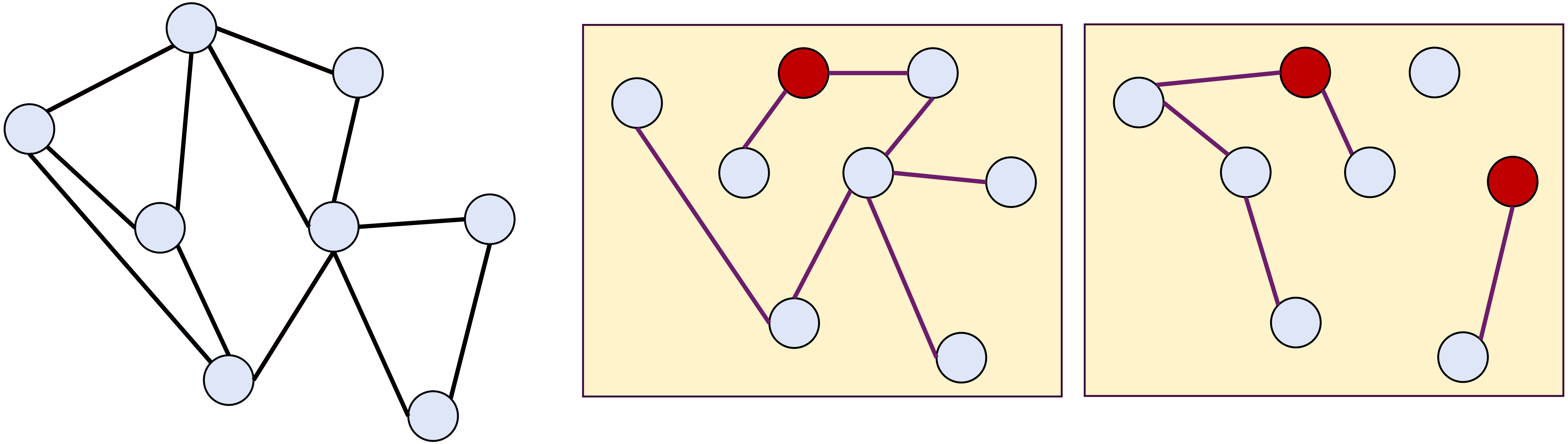

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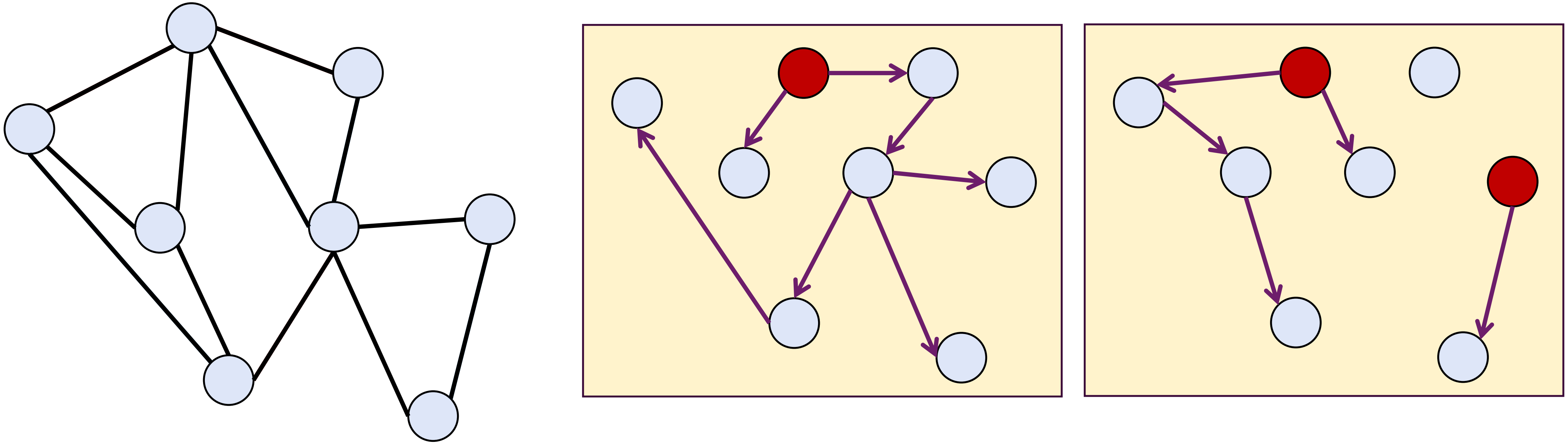

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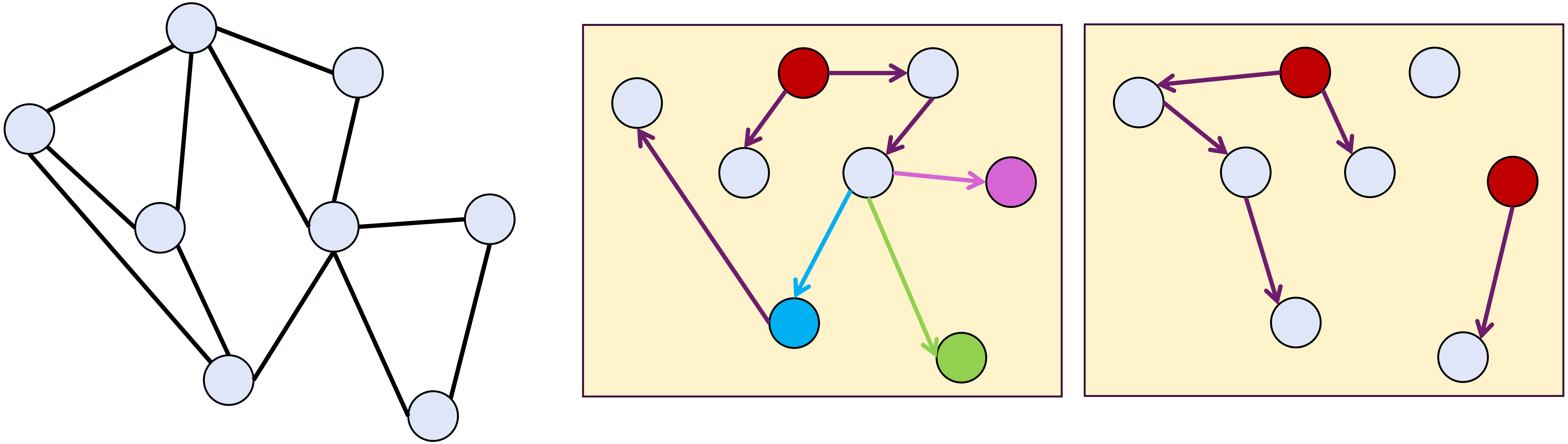

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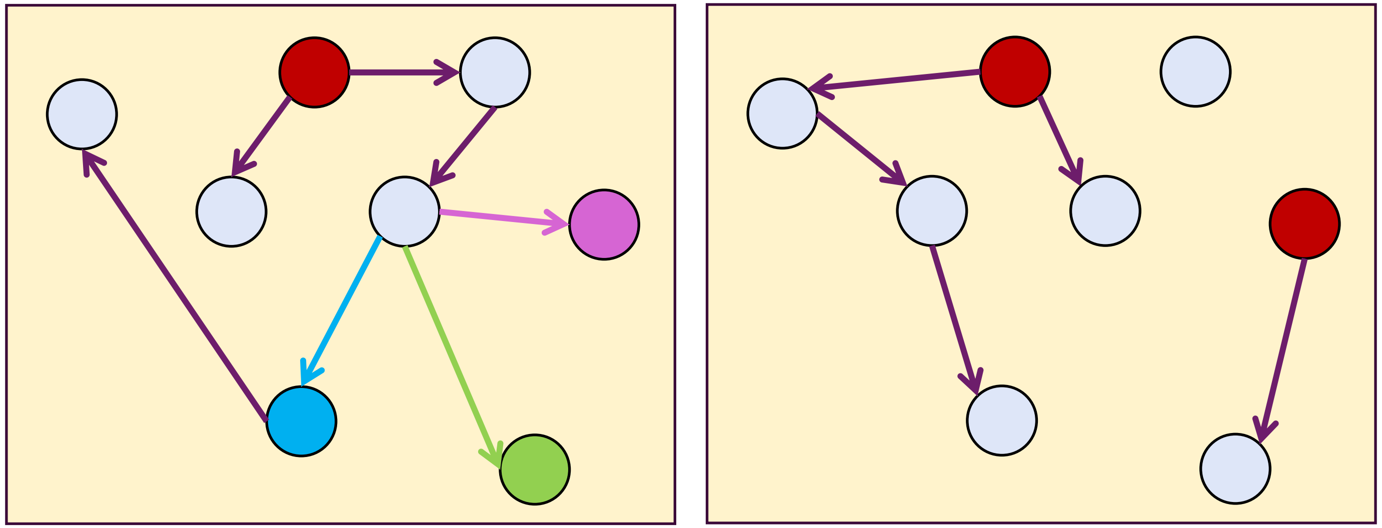

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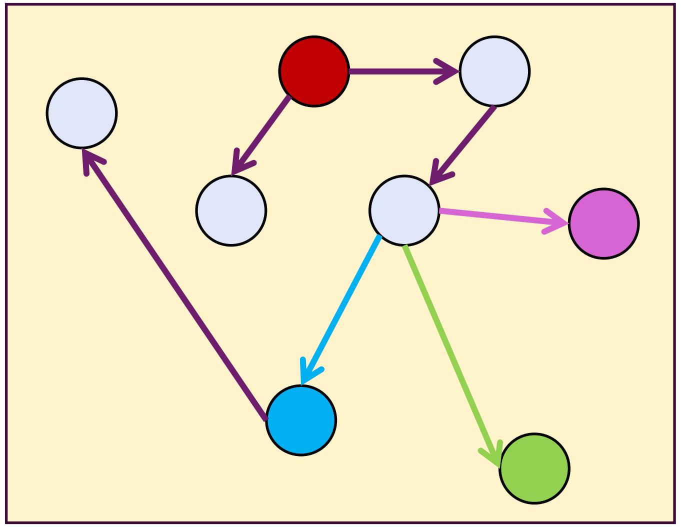
$$\sum_{(u,v)\in E} \min\left(\deg(u), \deg(v)\right)$$

$$\leq \sum_{1\leq i\leq \alpha} \sum_{e\in F_i} \deg\left(to(e)\right)$$

Follows because
$\deg\left(to(e)\right) \geq \min(\deg(u), \deg(v))$



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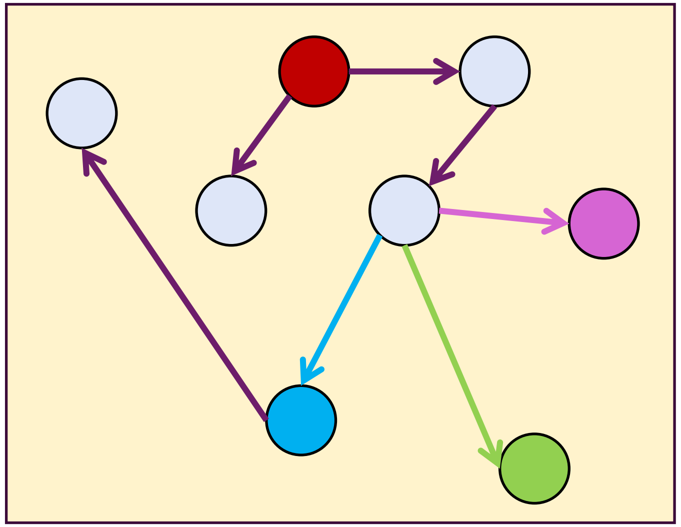
Since each vertex has at most one edge associated with it



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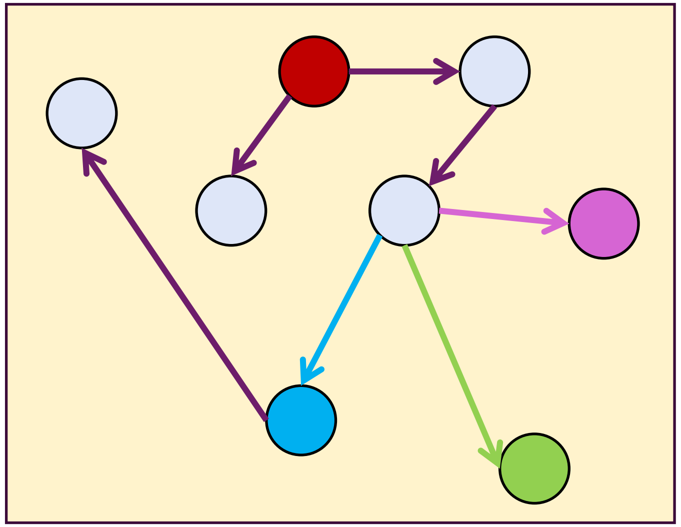
$$\leq \sum_{1\leq i\leq \alpha} \sum_{v\in V} \deg(v) = 2\alpha m$$

Since each vertex has at most one edge associated with it, the $\deg(to(e))$ of vertex $to(e)$ is counted at most once



Next Time

Bounded arboricity graphs in streaming model