Parallel Batch-Dynamic Algorithms for *k*-Core Decomposition and Related Problems

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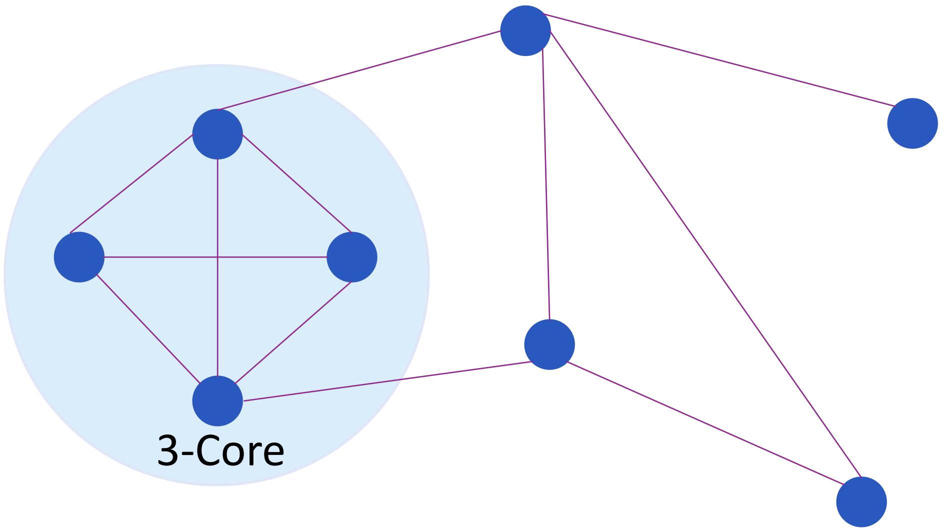
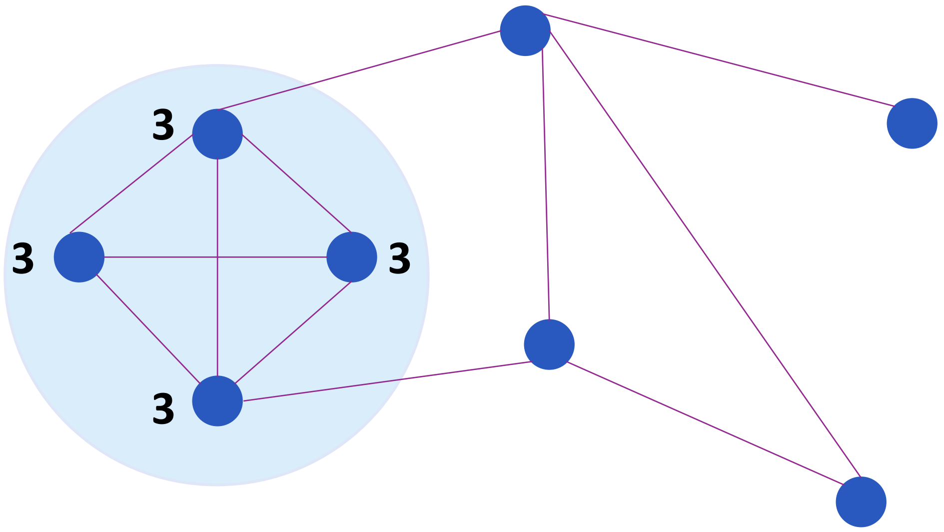
MIT

k-Core



k-Core Decomposition

Coreness or Core Number of Node v:
Maximum Core Value of a Core Containing v



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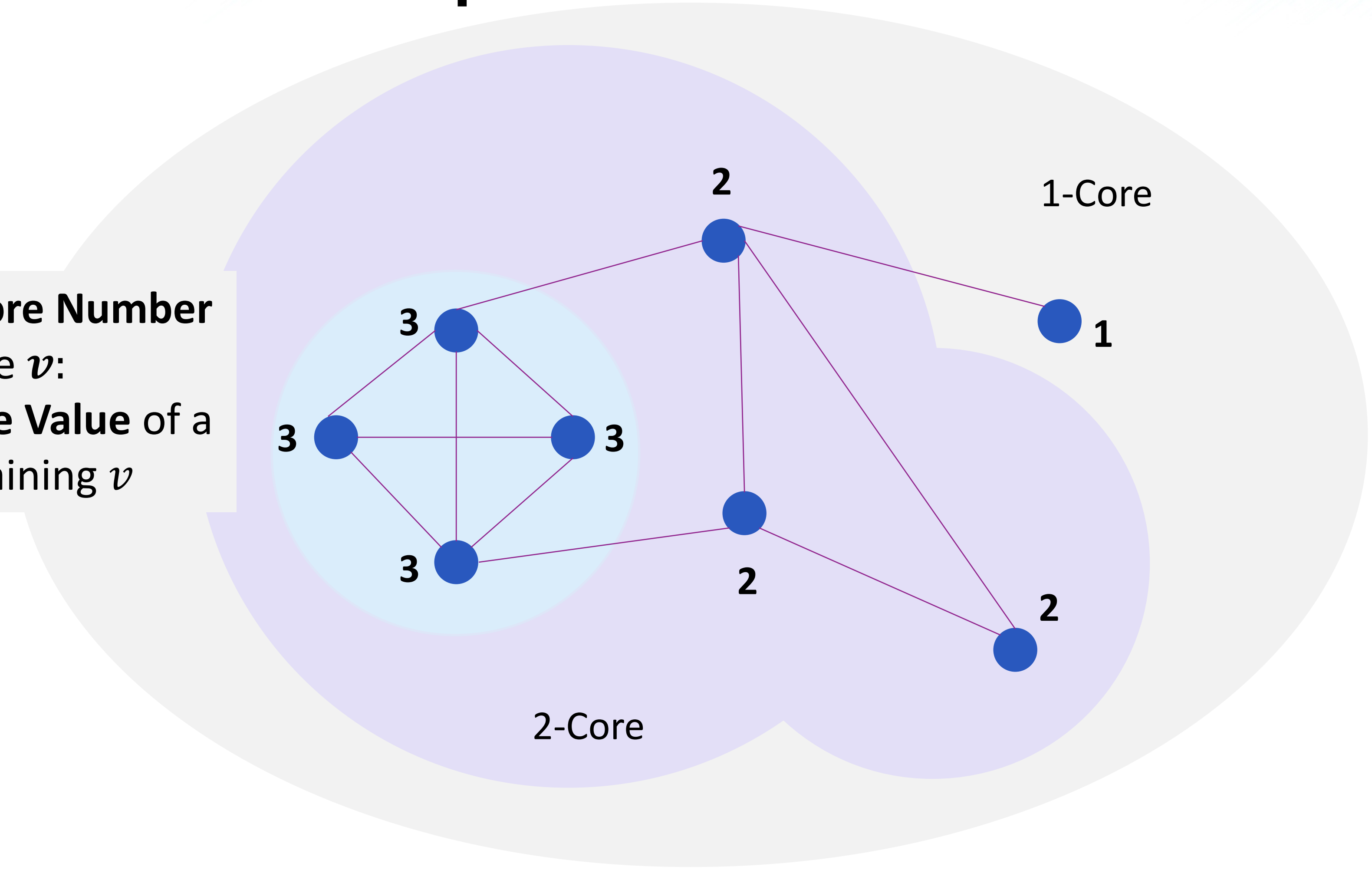
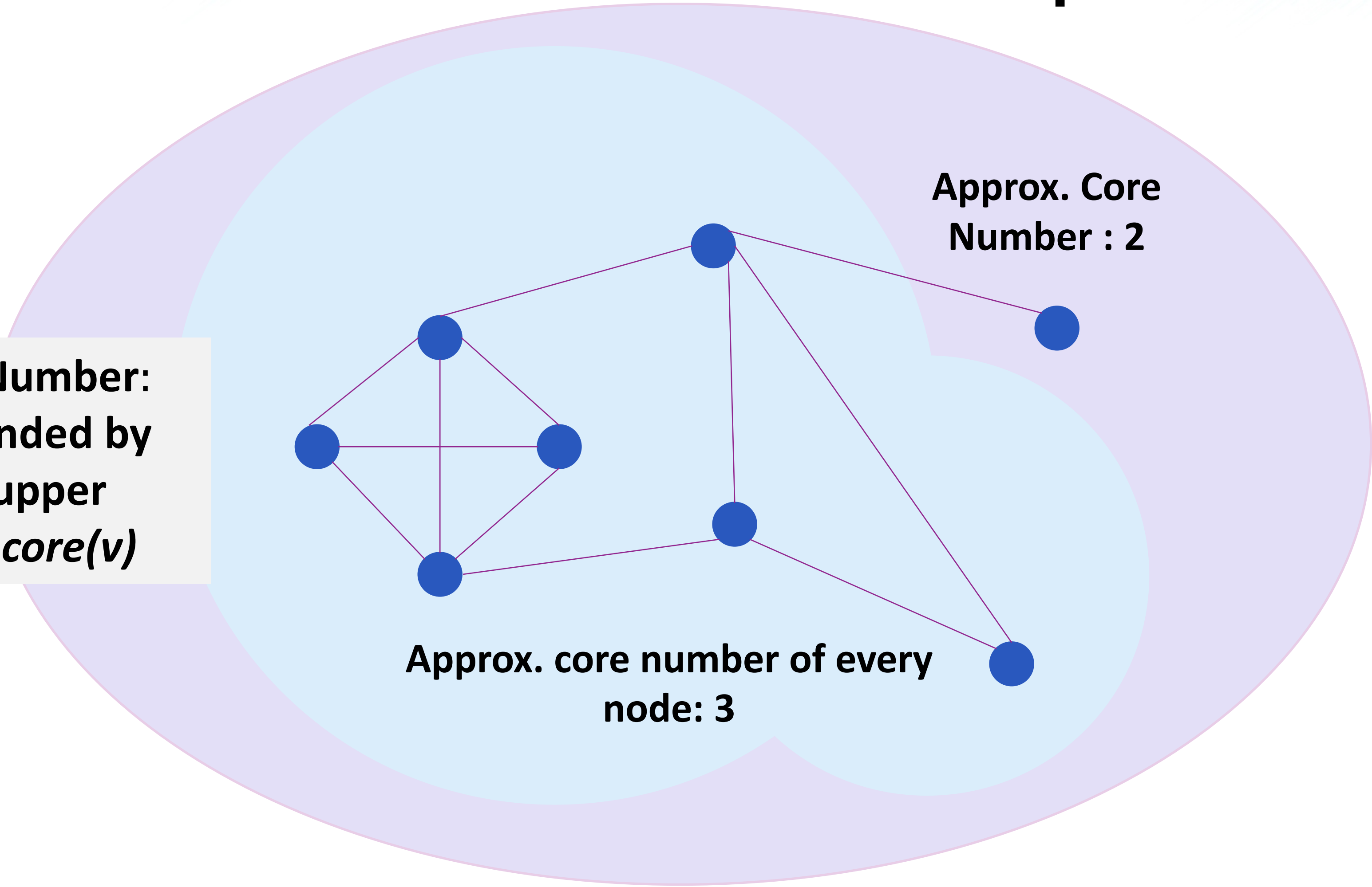


Approximate *k*-Core Decomposition

c-Approx. Core Number: Value **lower bounded by *core(v)/c*** and **upper bounded by *c * core(v)***



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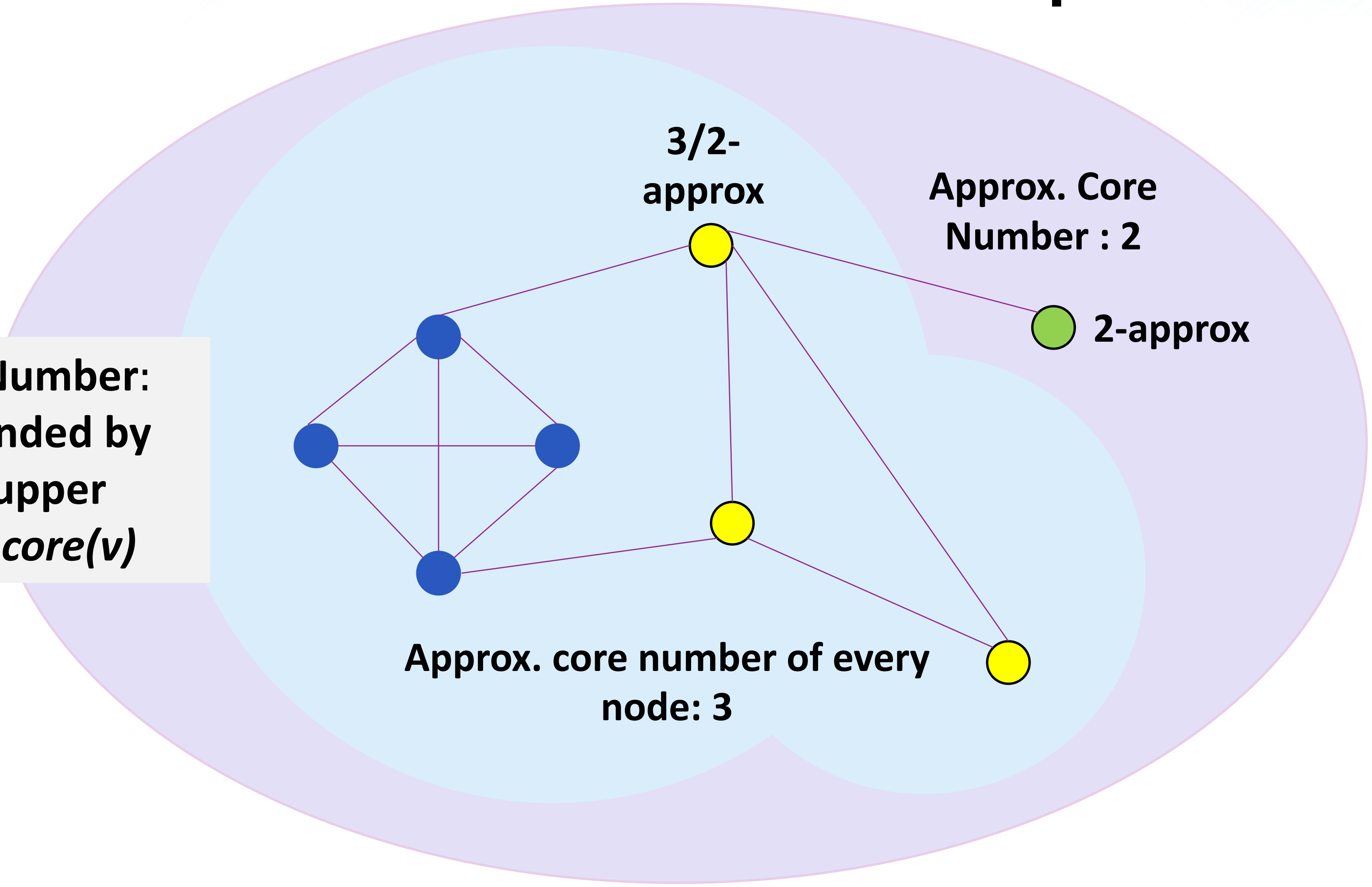

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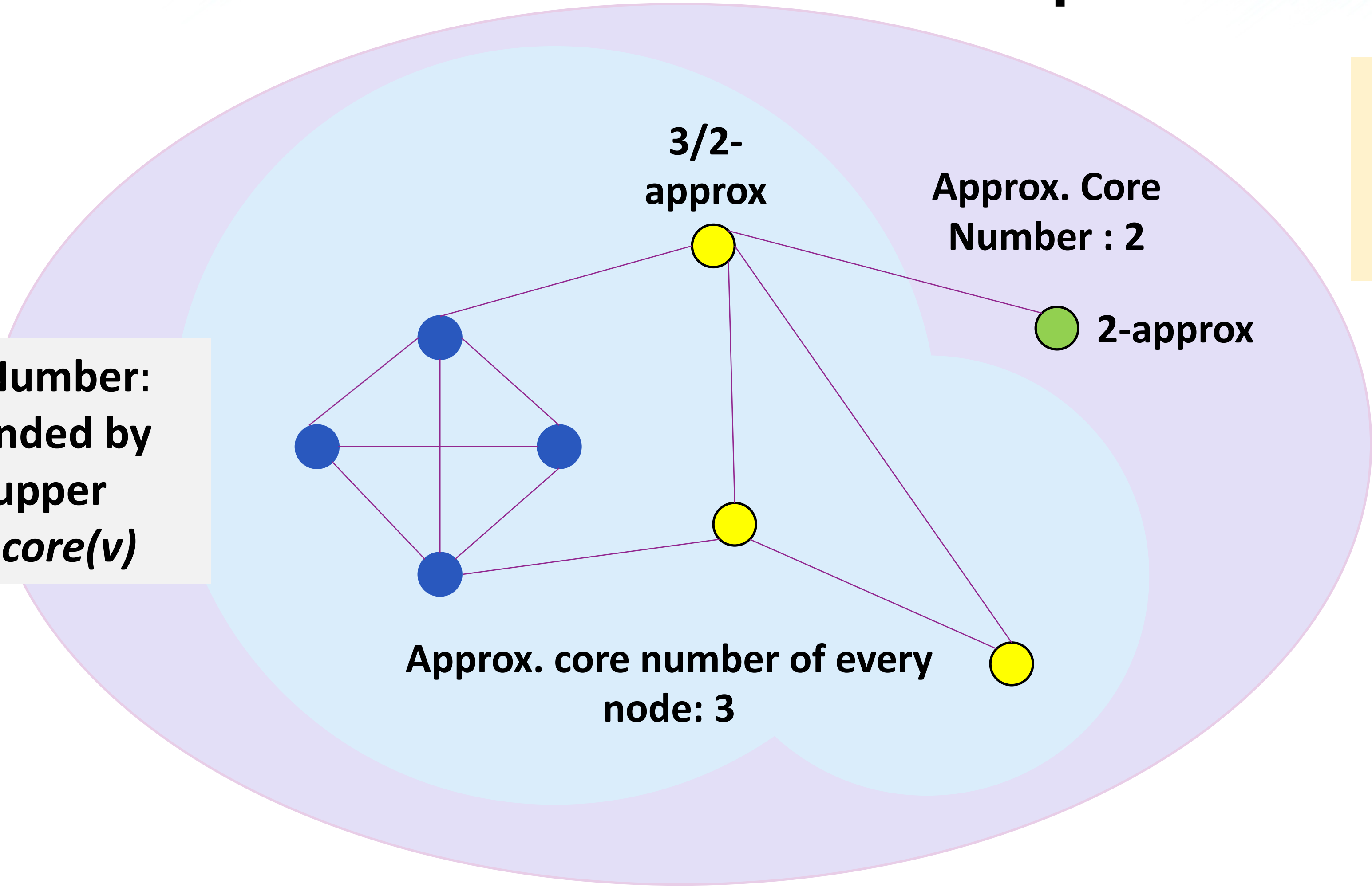
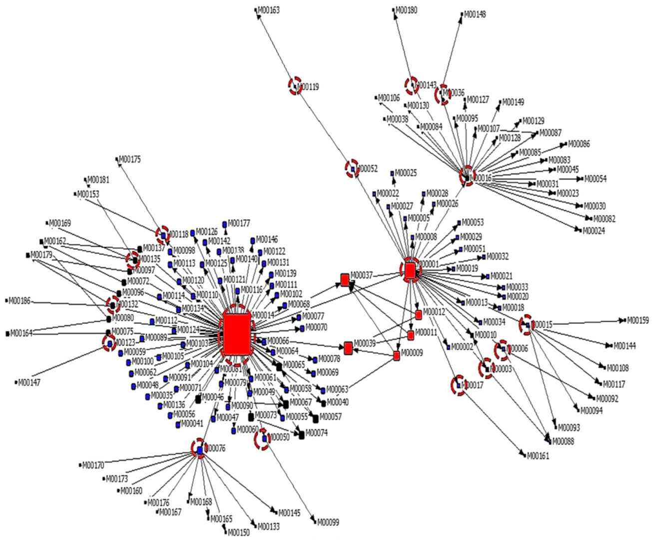
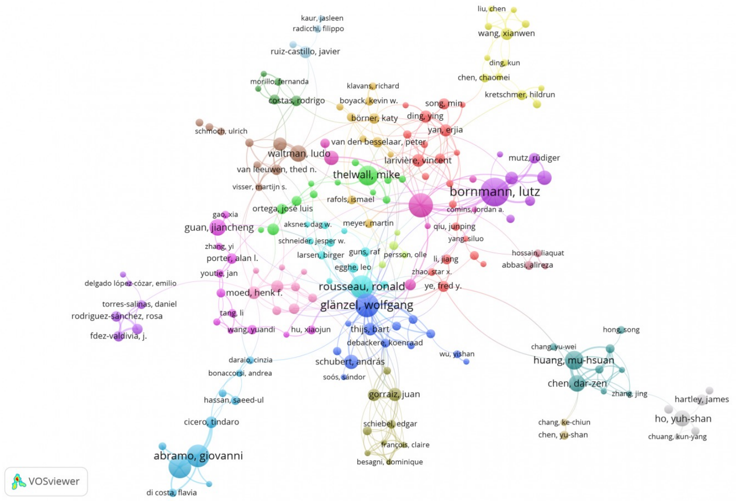


$(2+\varepsilon)$-approximations in this paper

Applications of *k*-Core Decomposition

- Graph clustering
- Community detection
- Graph visualizations
- Protein network analysis
- Modeling of disease spread
- Approximating network centrality measures
- Much interest in the machine learning, database, graph analytics, and other communities





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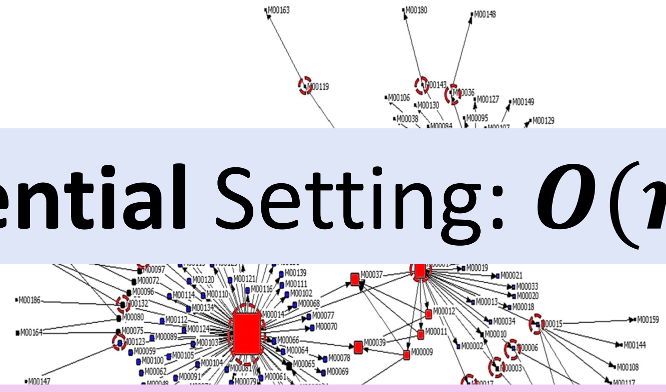
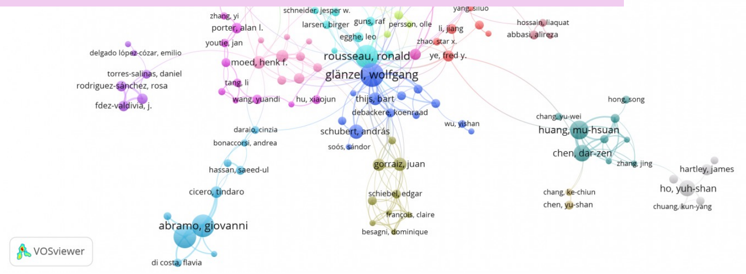


Static, Sequential Setting: $O(n)$ time

Billions or Even Trillions of Edges

Too Much Time to **Process Statically and Sequentially**



Large Graphs

facebook ~ 92.5 million edges

friendster ~ 1.8 billion edges

~ 2 billion edges

Common Crawl ~ 128 billion edges

Google ~ 6 trillion edges

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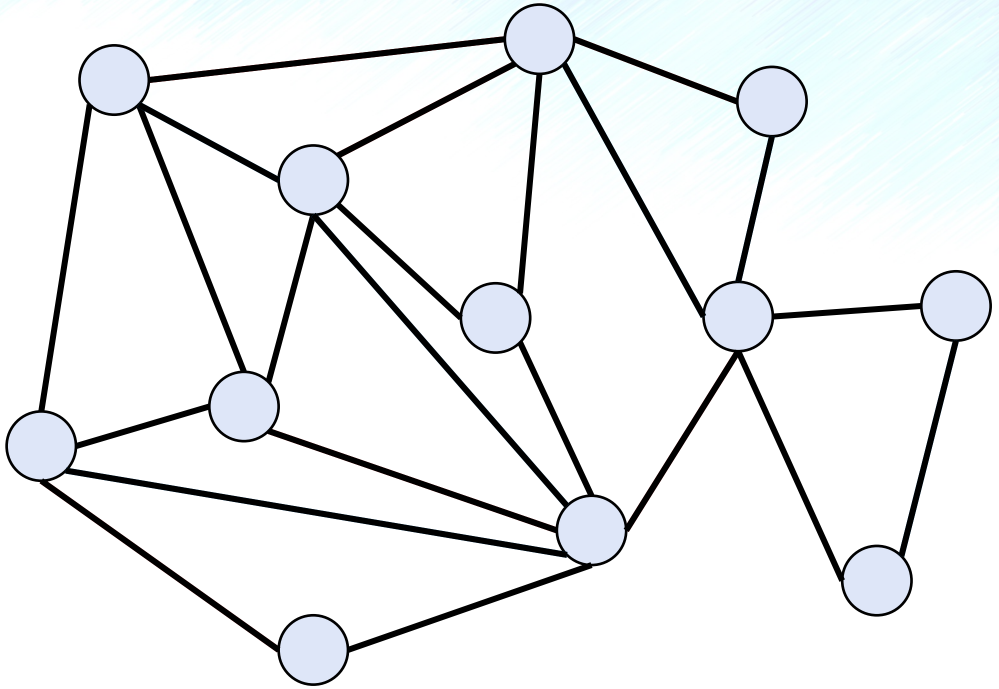


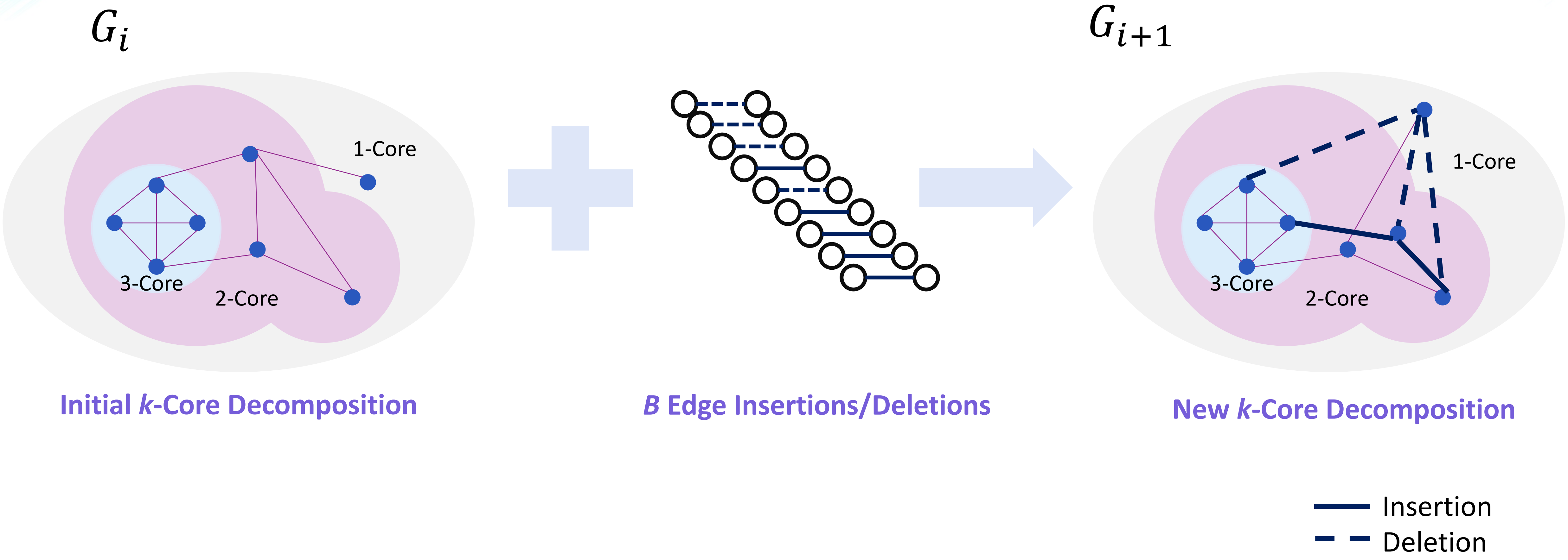
Graphs are rapidly changing:

- 3M emails/sec
- 486K WhatApp messages/sec
- 500M tweets/day
- 547K new websites/day

Work-Depth Model

- **Work:**
  - Total number of operations executed by algorithm
  - ***Work-efficient***: work asymptotically the same as *best-known sequential algorithm*
- **Depth:**
  - Longest chain of sequential dependencies in algorithm
- **Other Characteristics:**
  - Arbitrary forking
  - Concurrent read, concurrent write to the same shared memory

Batch-Dynamic Model Definition



Batch-Dynamic Graph Algorithms

- Triangle counting [Ediger et al. ‘10, Makkar et al. ’17, Dhulipala et al. ‘20]
- Euler Tour Trees [Tseng et al. ‘19]
- Connected Components [Ferragina and Lucio ‘94, McColl et al. ‘13; Acar et al. ’19, Nowicki and Onak ‘21]
- Rake-Compress Trees [Acar et al. ‘20]
- Incremental Minimum Spanning Trees [Anderson et al. ‘20]
- Minimum Spanning Forest/Graph Clustering [Nowiki and Onak ‘21, Tseng et al. ‘22]
- Graph Connectivity [Dhulipala et al. ‘20]
- Maximal Matching [Nowicki and Onak ‘21]

Why Approximate *k*-Core Decomposition

- **Dynamic exact *k*-core decomposition:**
  - **$\Omega(n)$ work, $\Omega(n)$ depth, parallel** [Aridhi et al '16, Gabert et al. '21, Hua et al. '20, Jin et al. '18, Wang '17]
  - One update can cause **$\Omega(n)$ coreness changes**

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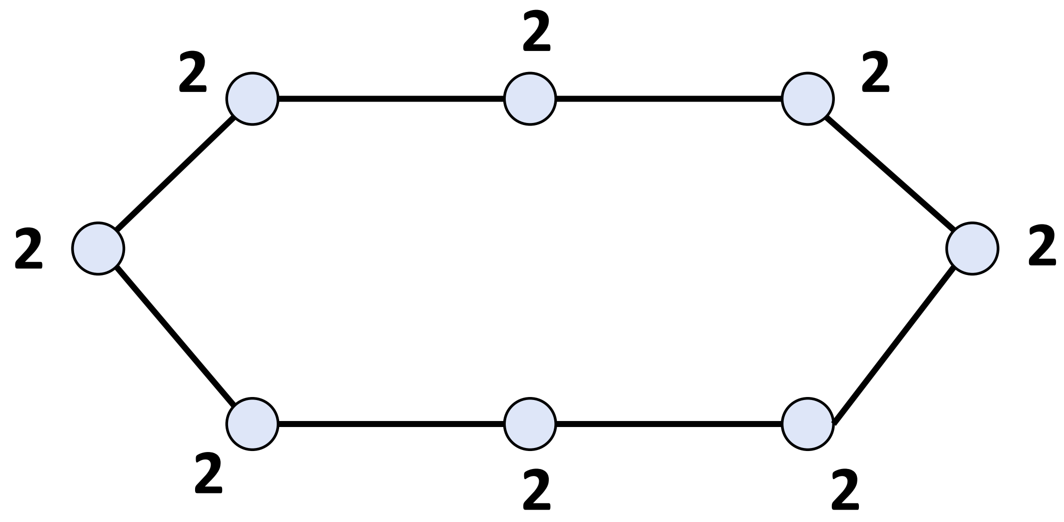

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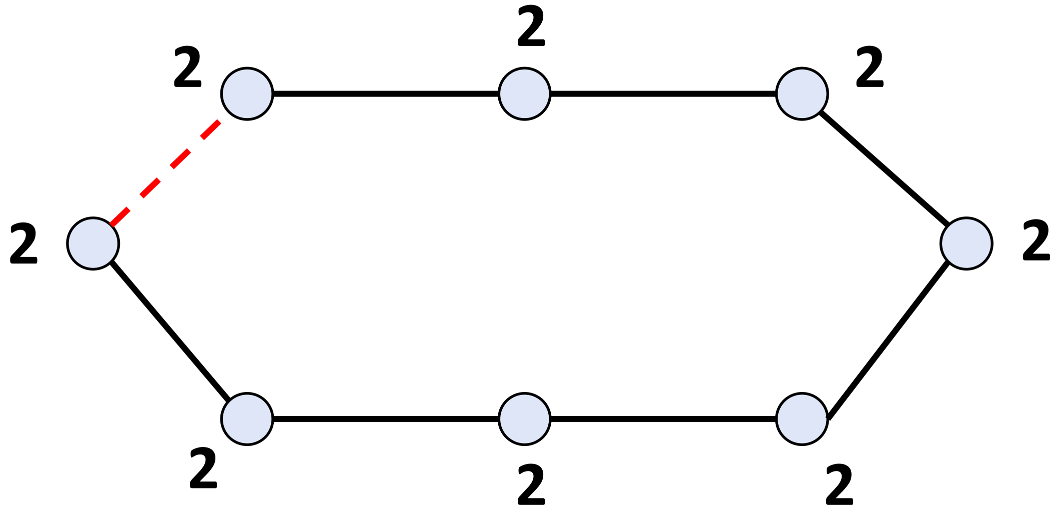

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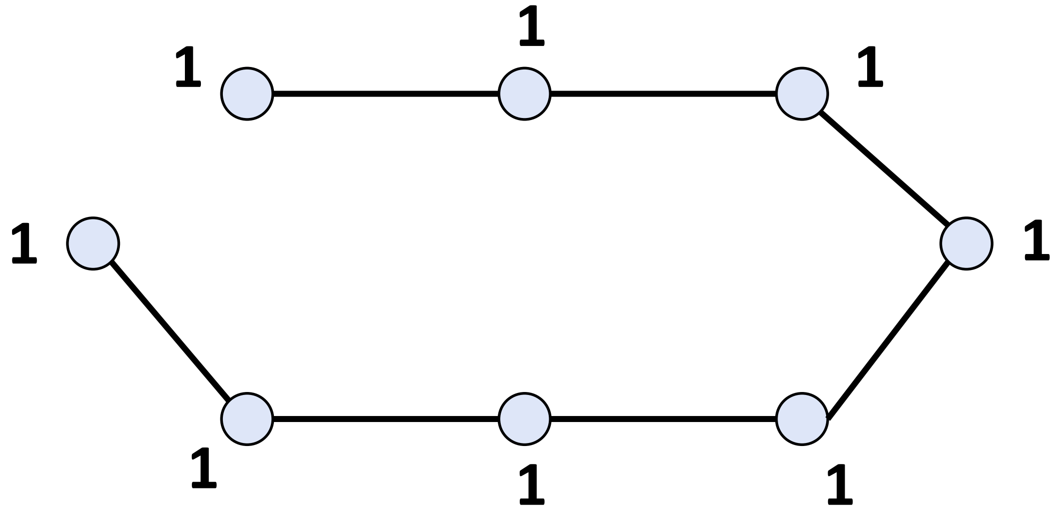

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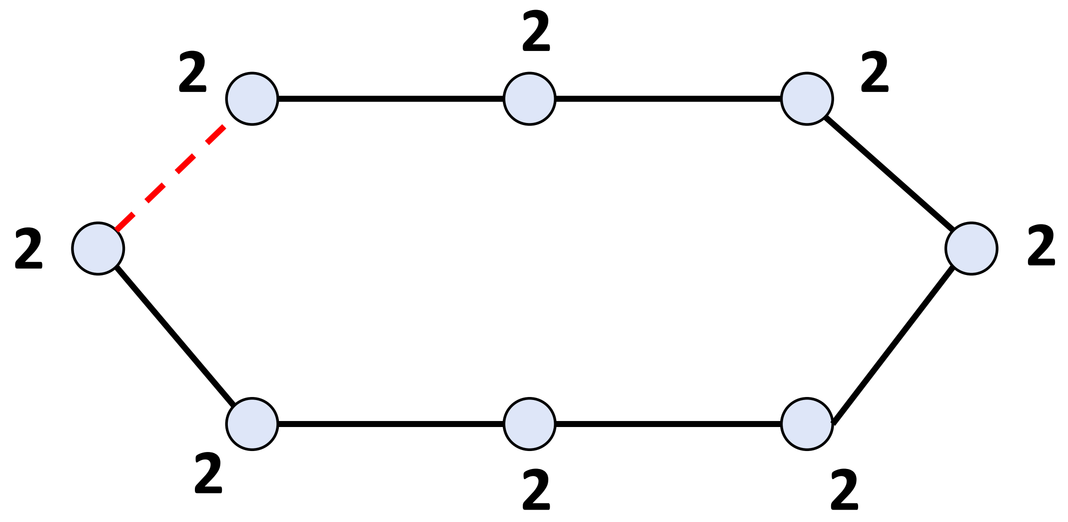

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  - $O(\log^2 n)$ **time amortized, sequential,** $(2+\varepsilon)$**-approximation** [Sun et al. '20]
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  - **Threshold peeling** procedure

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Does not use parallelism

One update at a time

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Caveat: amortized $O(\log^2 n)$ depth,
worst-case $\Omega(n)$ depth

Want: worst-case $\text{poly}(\log n)$ depth

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Batch Dynamic *k*-Core Decomposition **+ Others!**

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Static *k*-Core Decomposition
Low Out-Degree Orientation
Maximal Matching
Clique Counting
Vertex Coloring

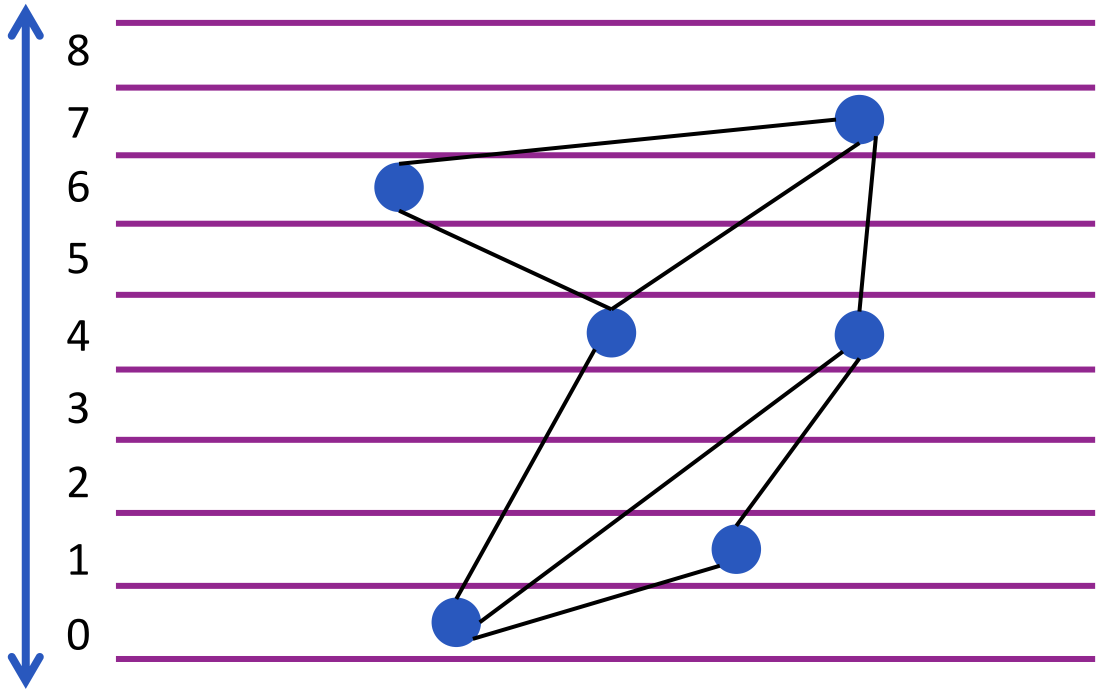
Sequential Level Data Structures for Dynamic Problems

- Maximal Matching [Baswana-Gupta-Sen 18, Solomon ‘16]
- $(\Delta + 1)$-Coloring [Bhattacharya-Chakrabarty-Henzinger-Nanongkai ‘18, Bhattacharya-Grandoni-Kulkarni-**L**-Solomon ‘19]
- Clustering [Wulff-Nilsen ‘12]
- Low out-degree orientation [Solomon-Wein 20, Henzinger-Neumann-Weiss ‘20]
- Densest subgraph [Bhattacharya-Henzinger-Nanongkai-Tsourakakis ‘15]

Sequential Level Data Structure (LDS)

$O(\log^2 n)$

Vertices partitioned into levels



Bhattacharya-Henzinger-Nanongkai-Tsourakakis STOC 2015
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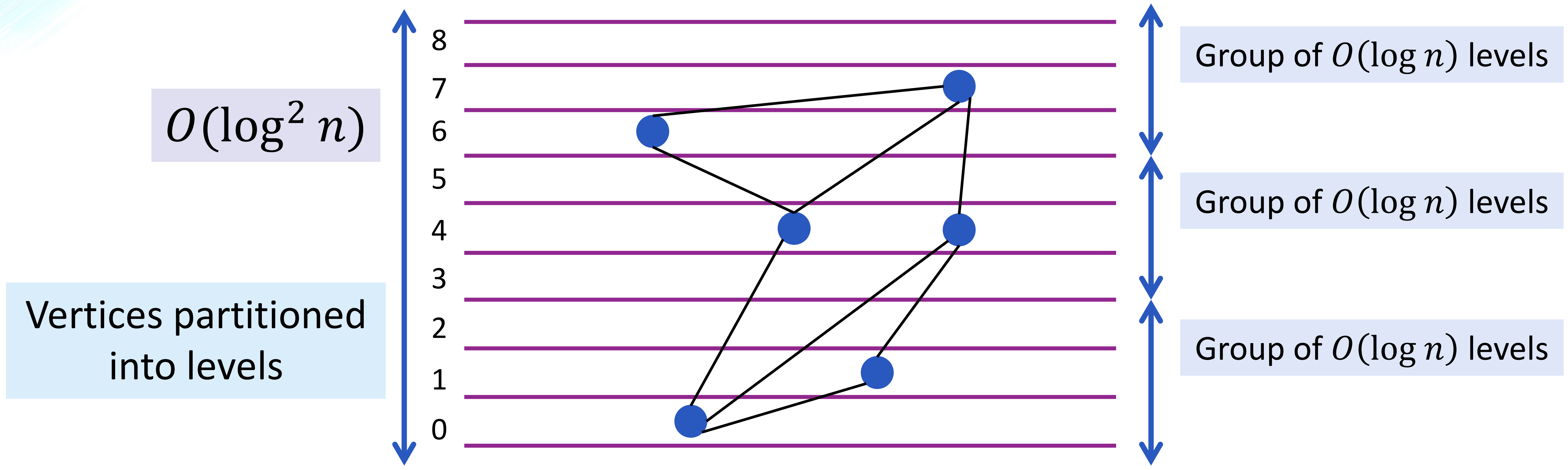

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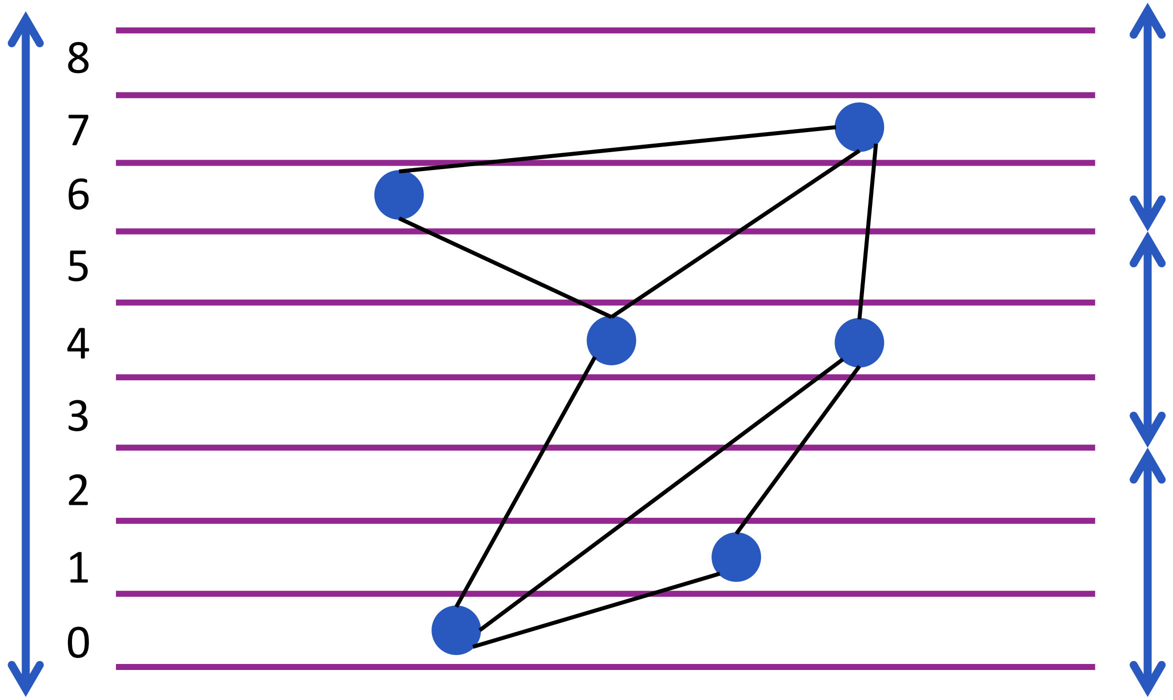

Group of $O(\log n)$ levels

Cut-off: $(1+\epsilon)^i$

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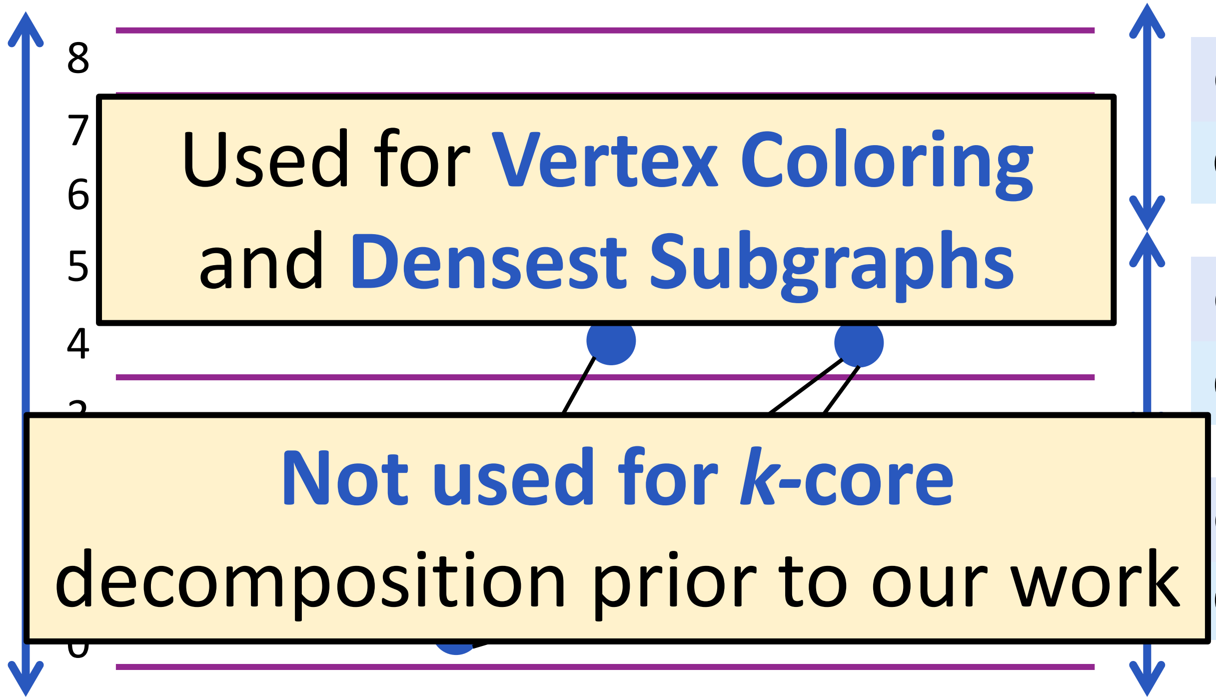

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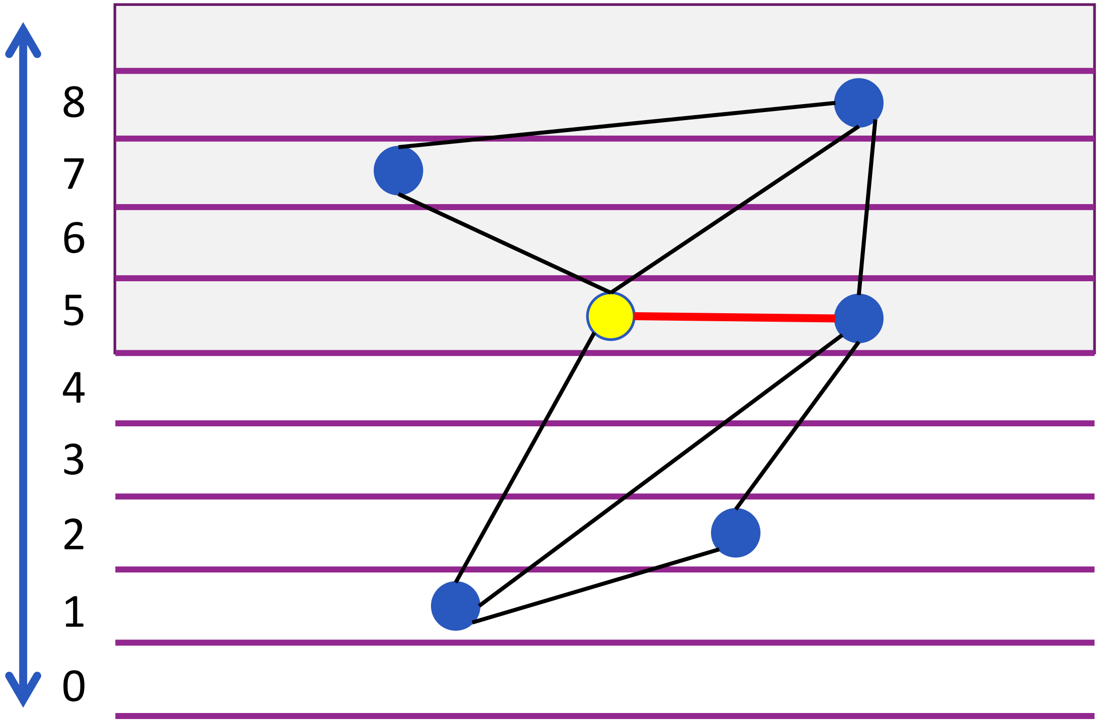

neighbors: $> 2.1(1+\epsilon)^i$

= edge insertion

Bhattacharya-Henzinger-Nanongkai-Tsourakakis STOC 2015 and Henzinger-Neumann-Weiss 2020

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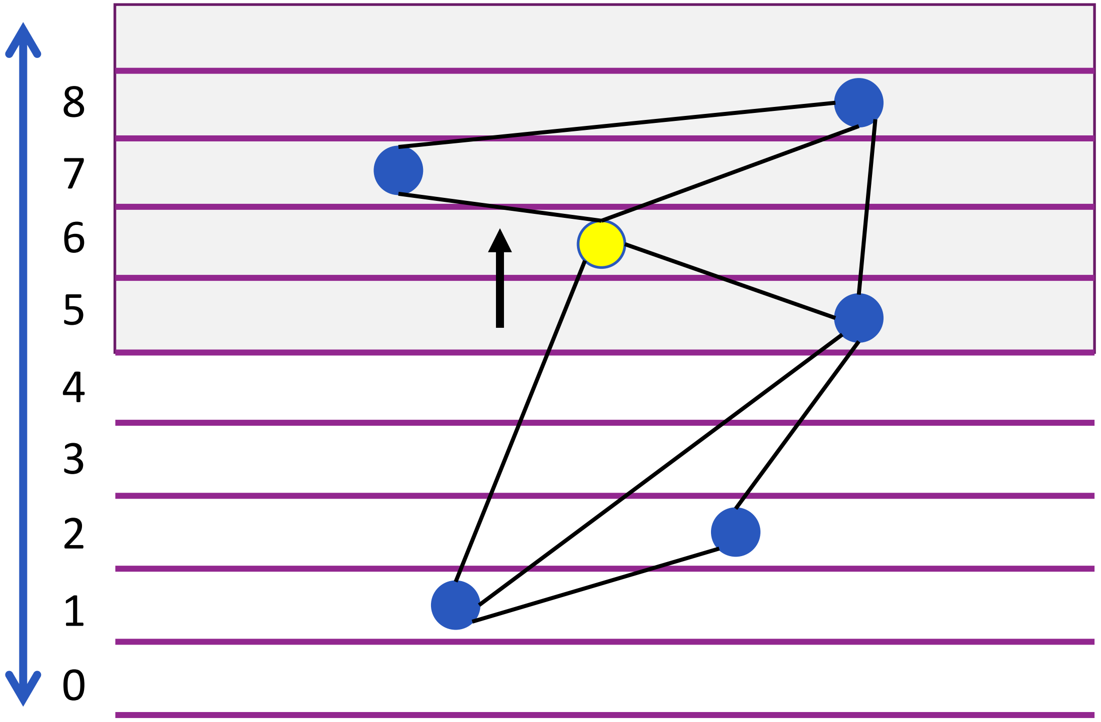

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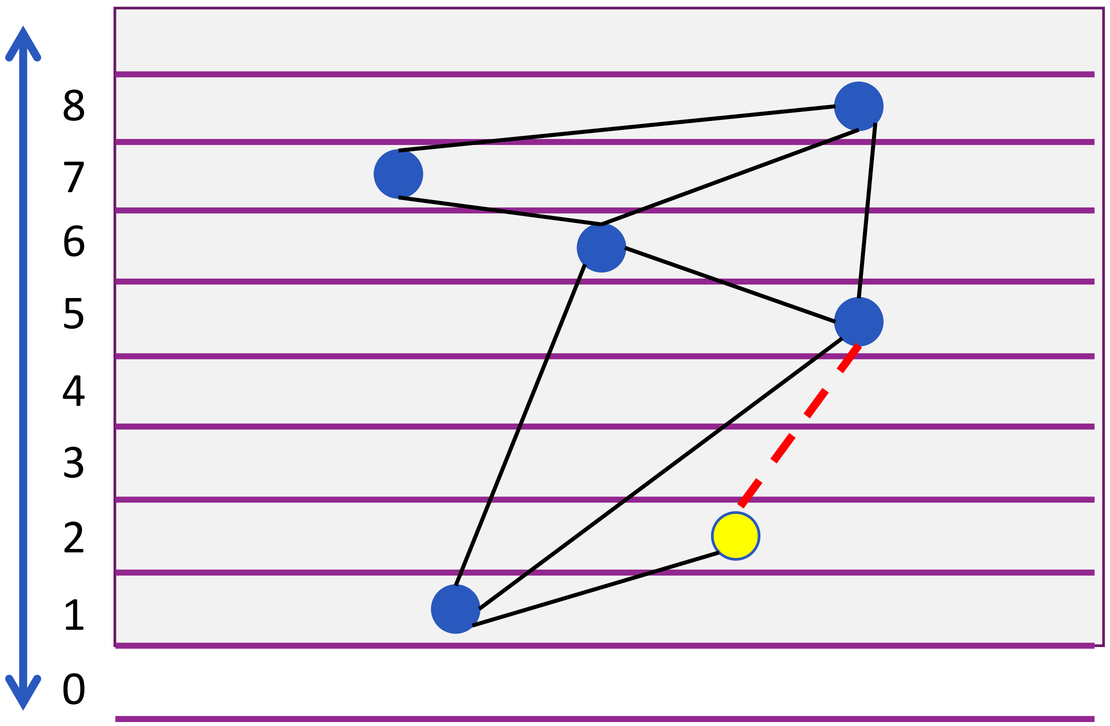

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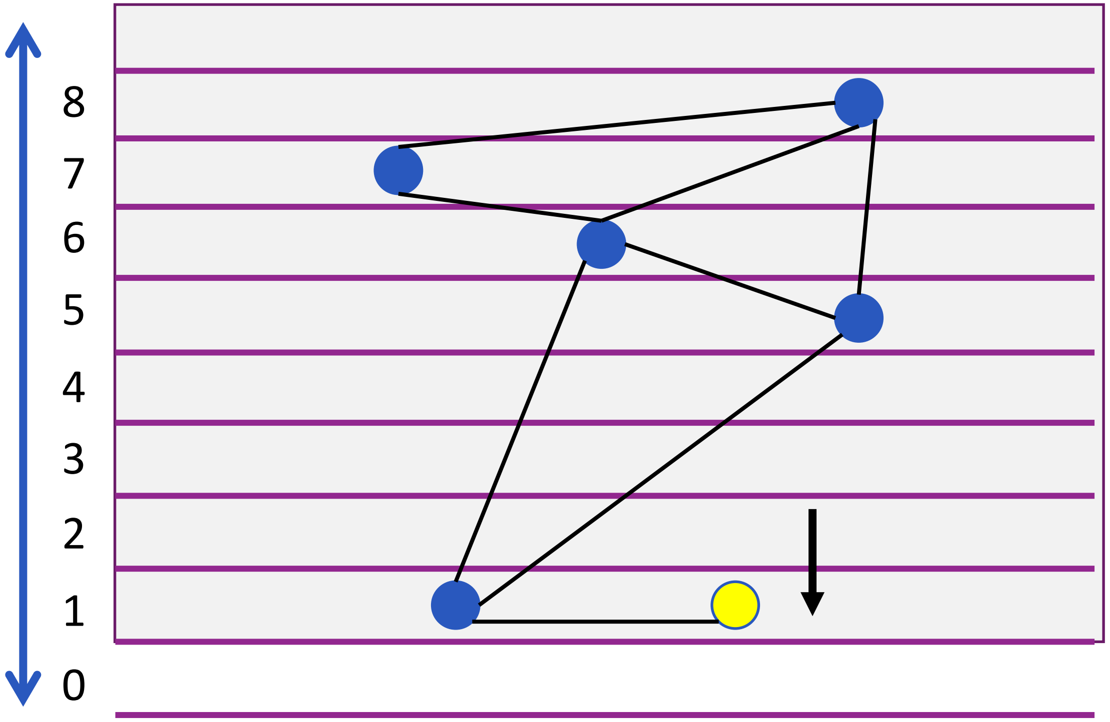

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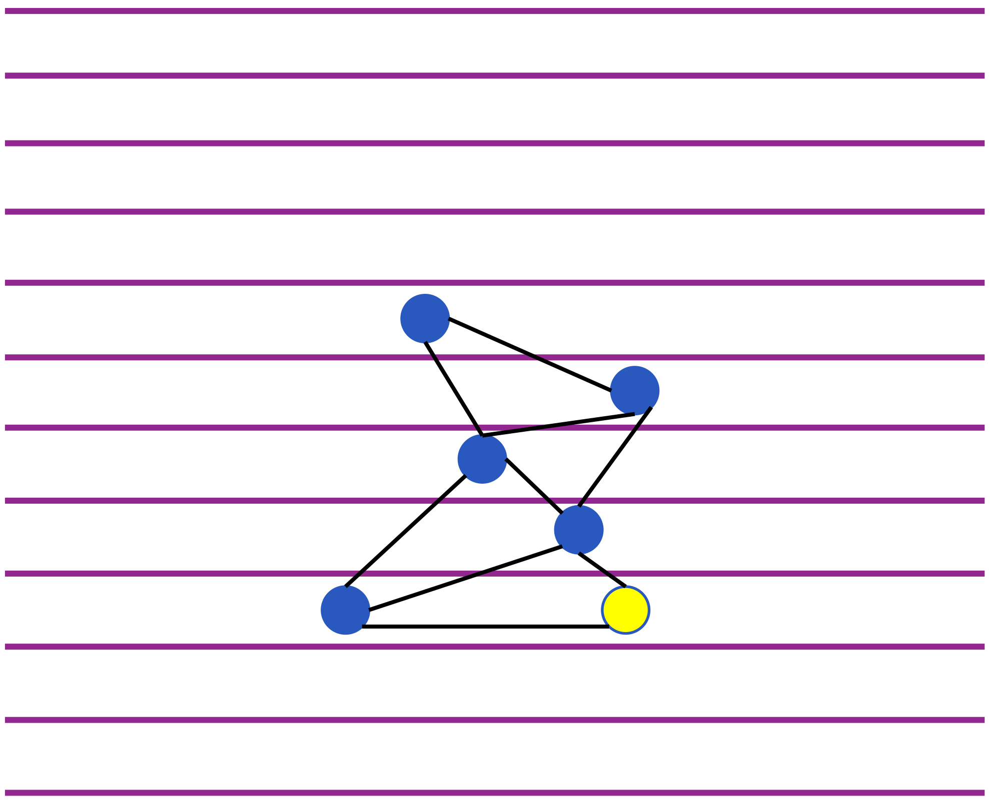
Large sequential dependencies



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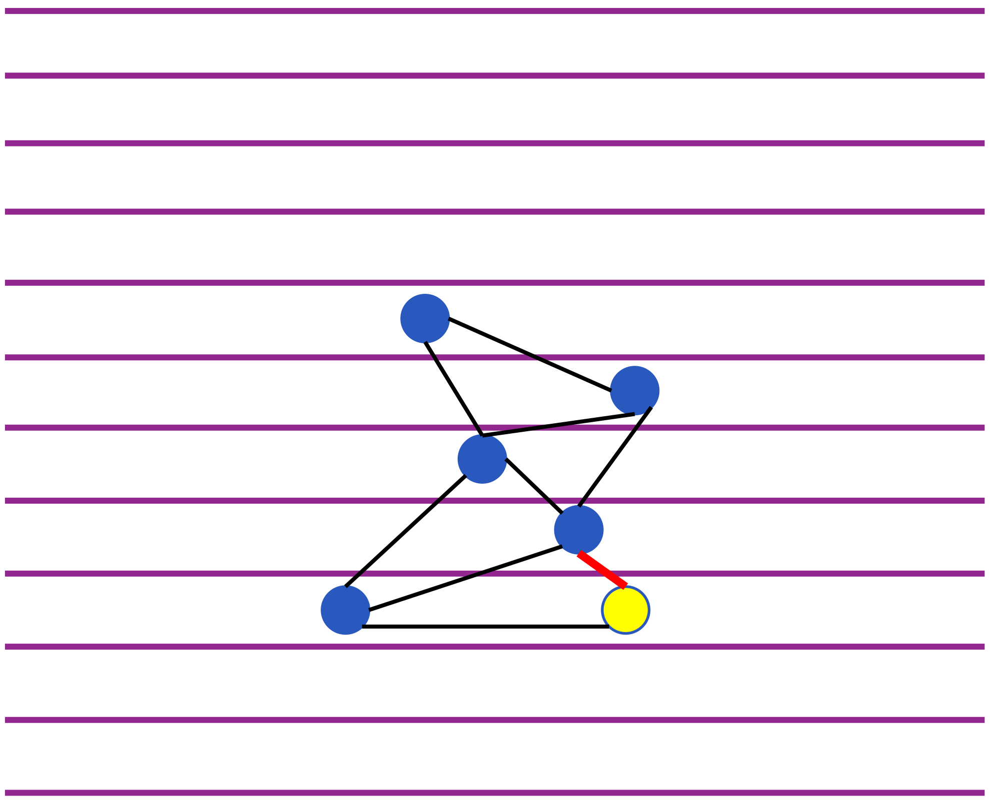

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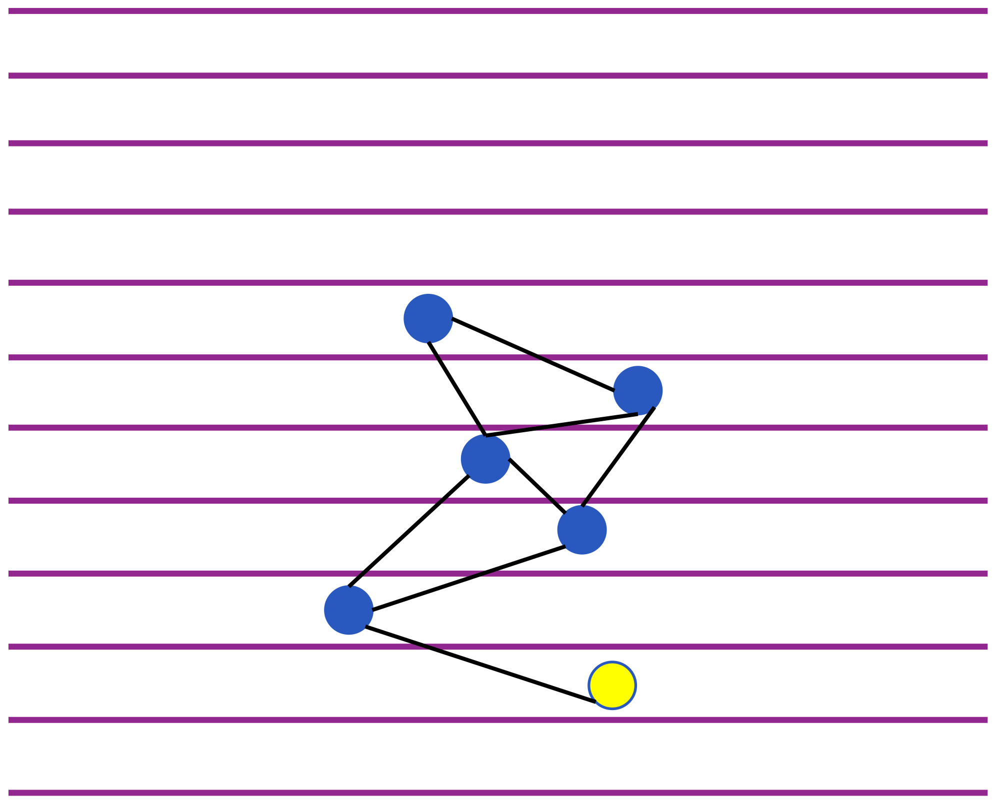

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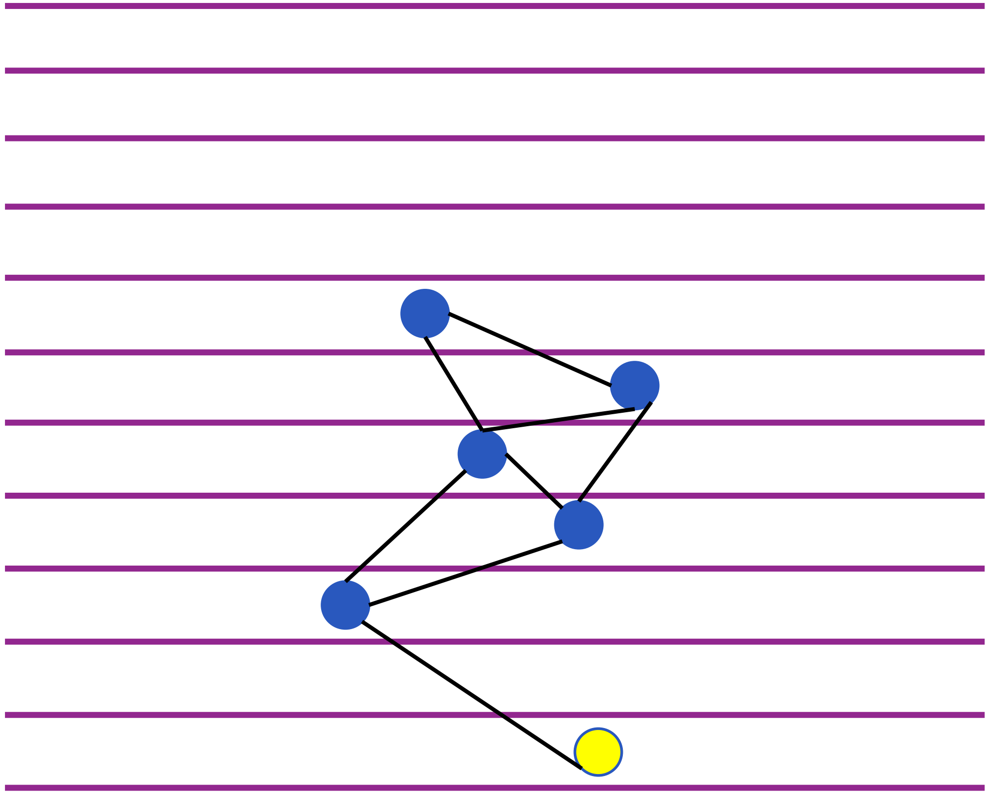

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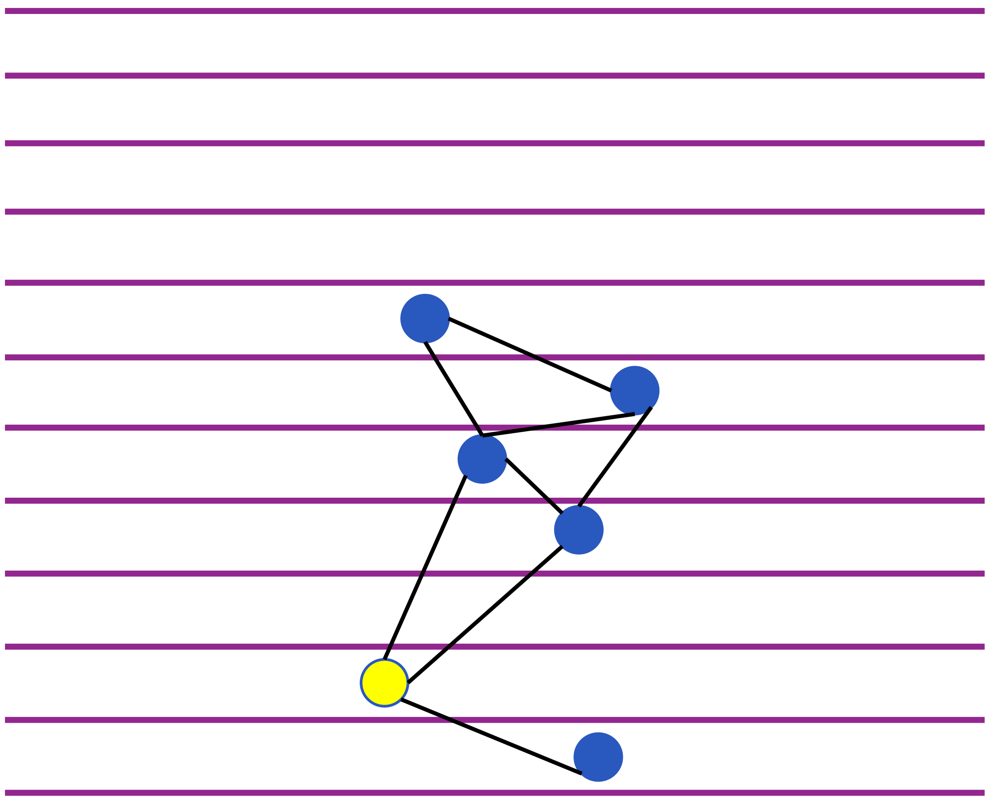

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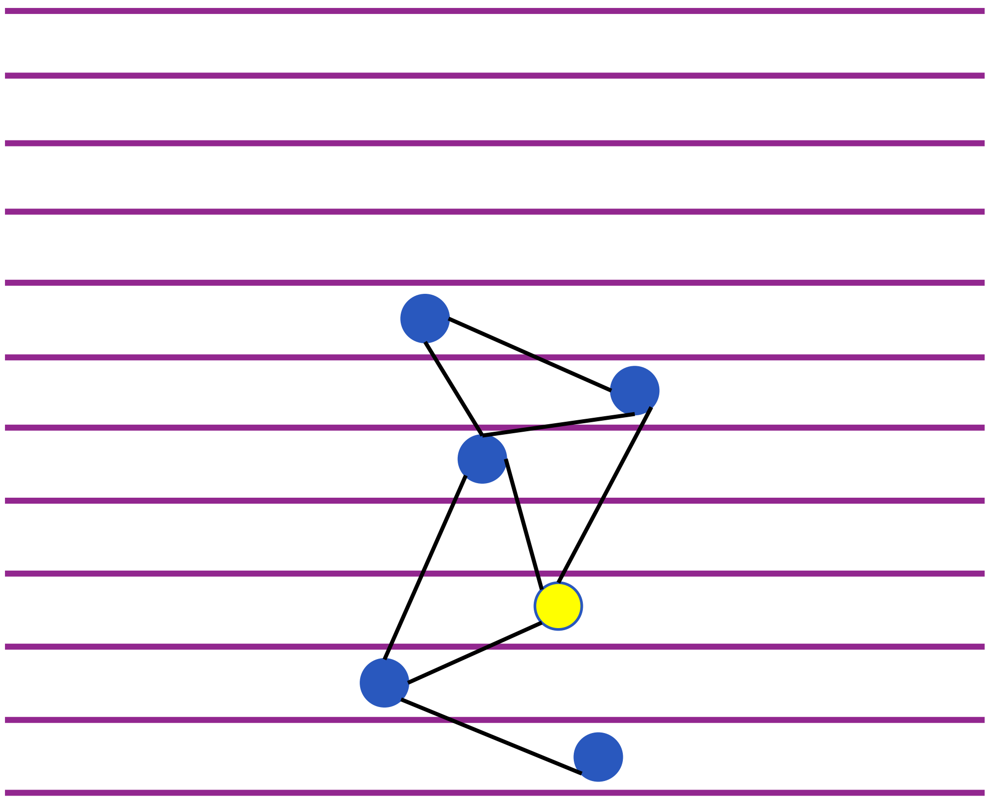

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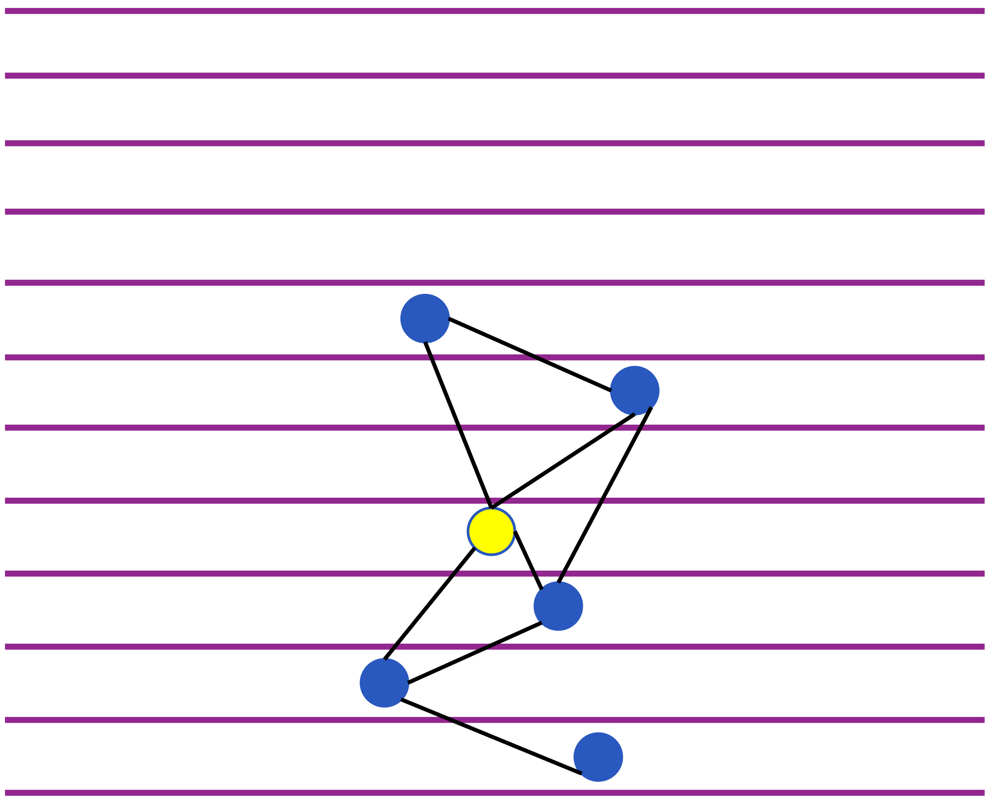

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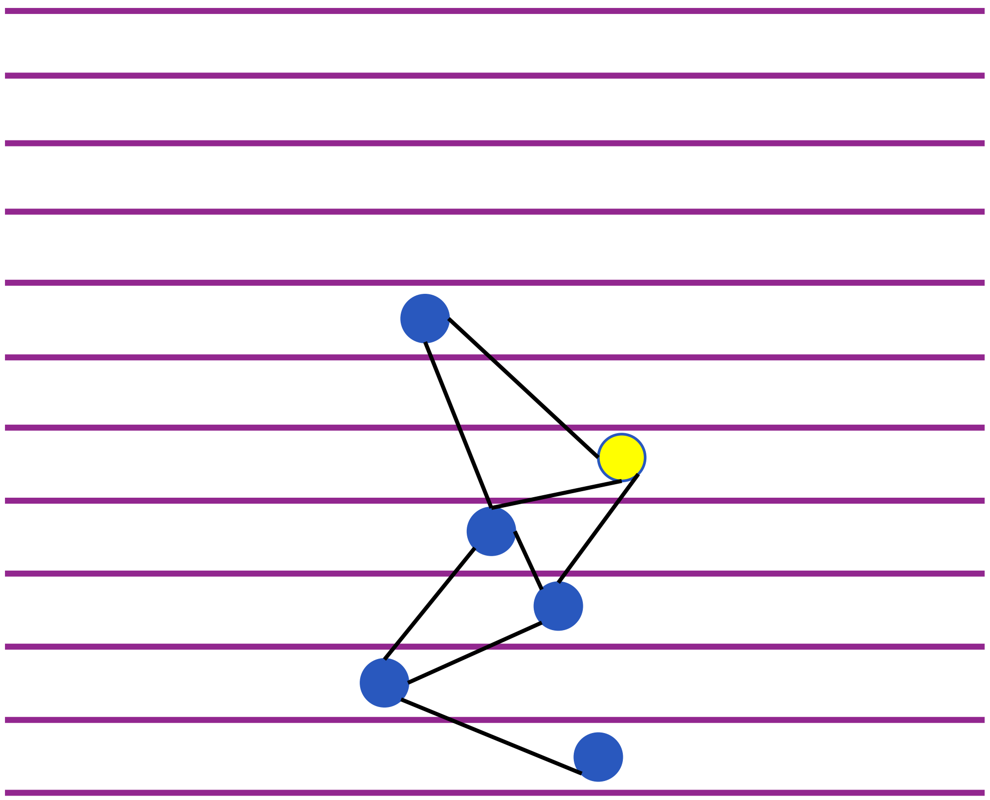

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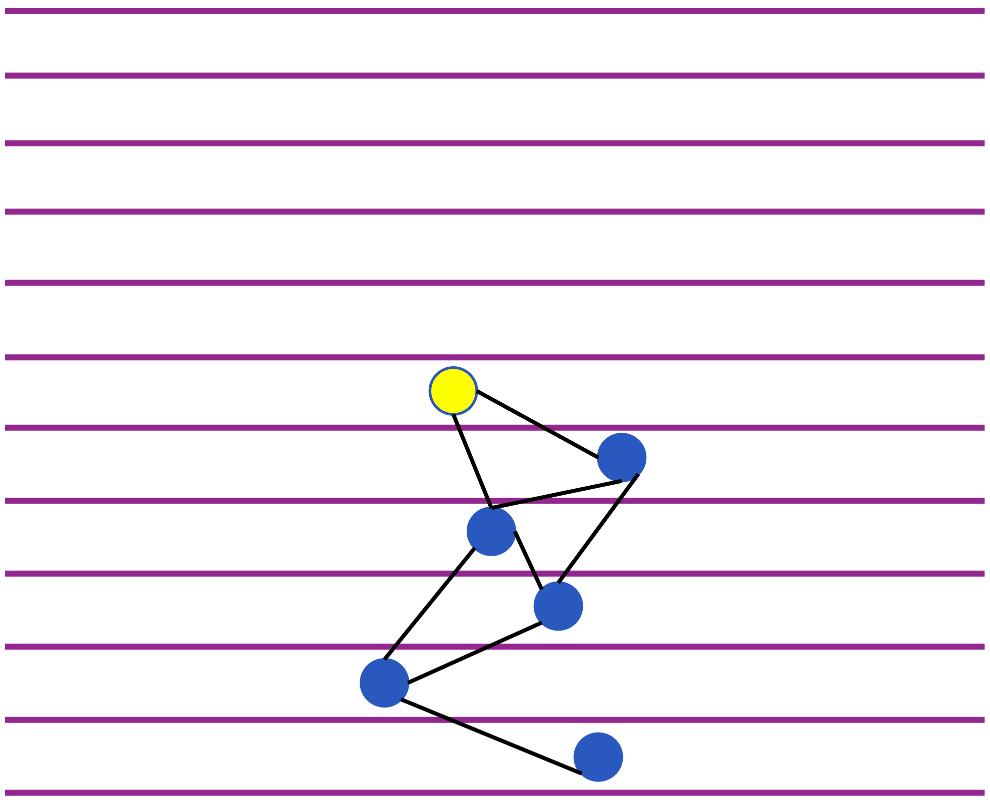

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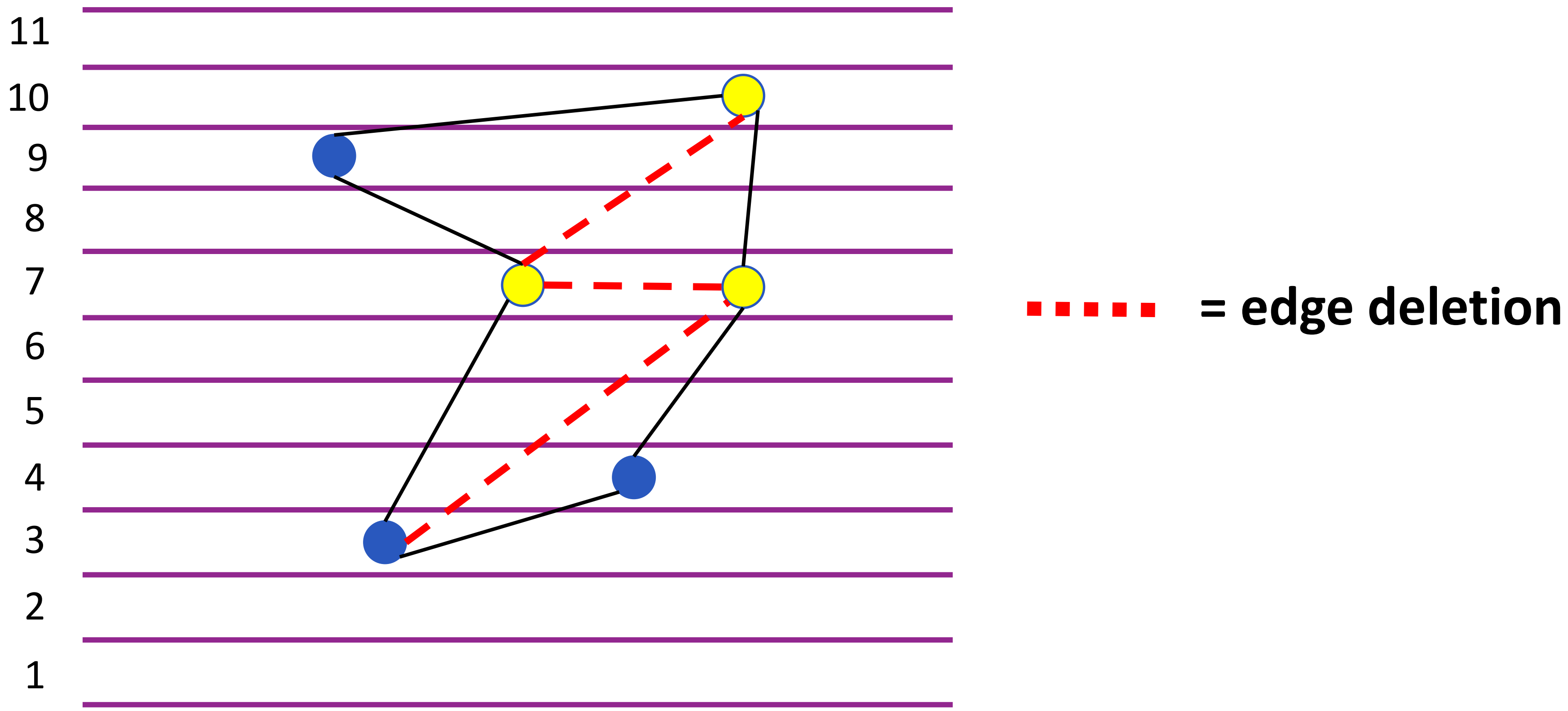
Our Parallel Batch-Dynamic Level Data Structure (PLDS)

Deletions



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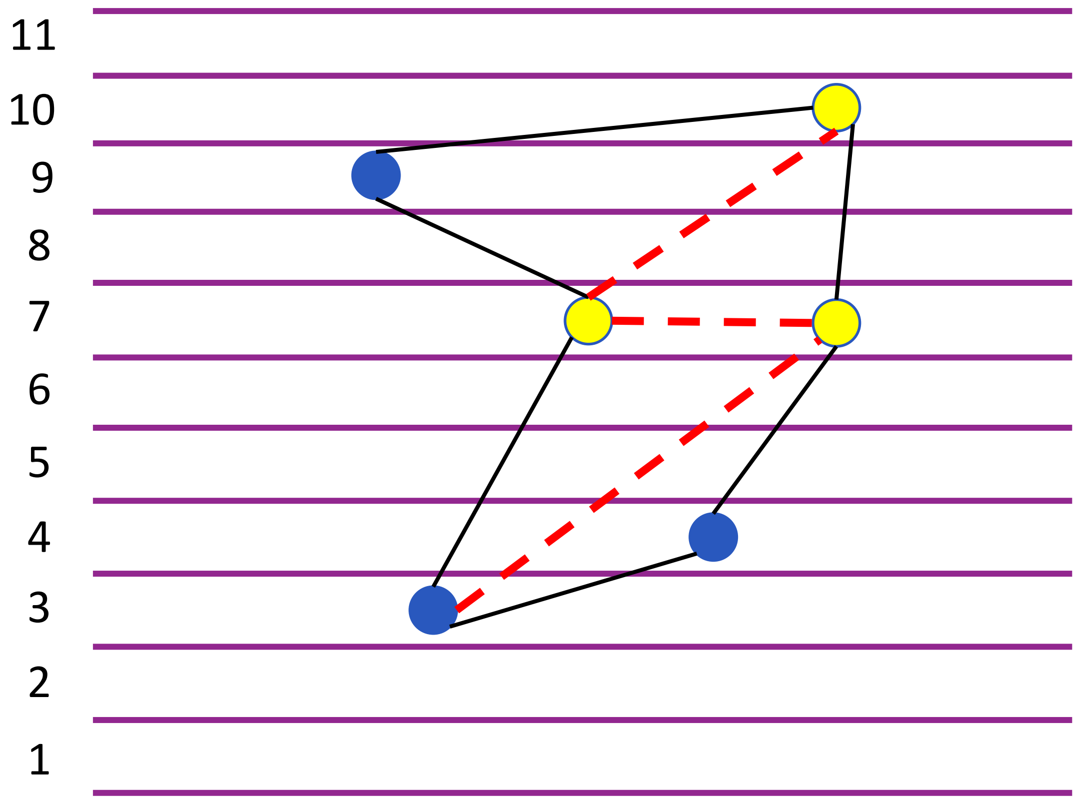

= edge deletion

Only lower bound cutoff, $(1+\epsilon)^i$, ever violated.

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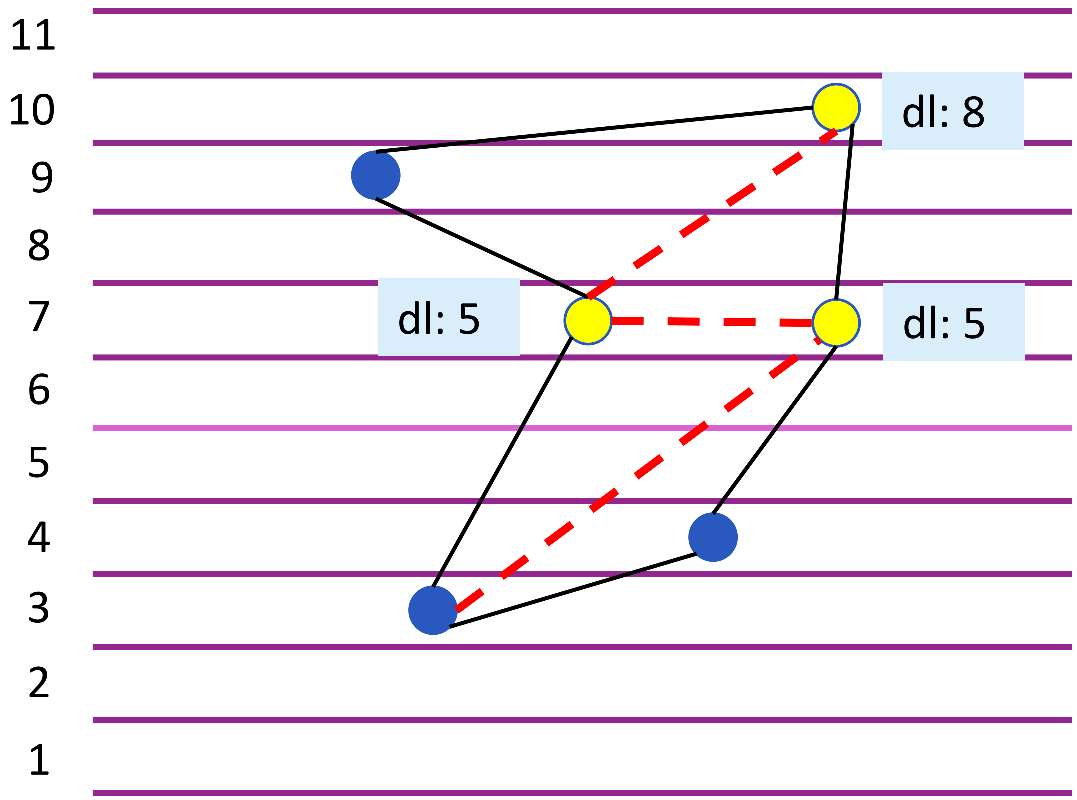
Calculate *desire-level*: **closest level that satisfies cutoffs**



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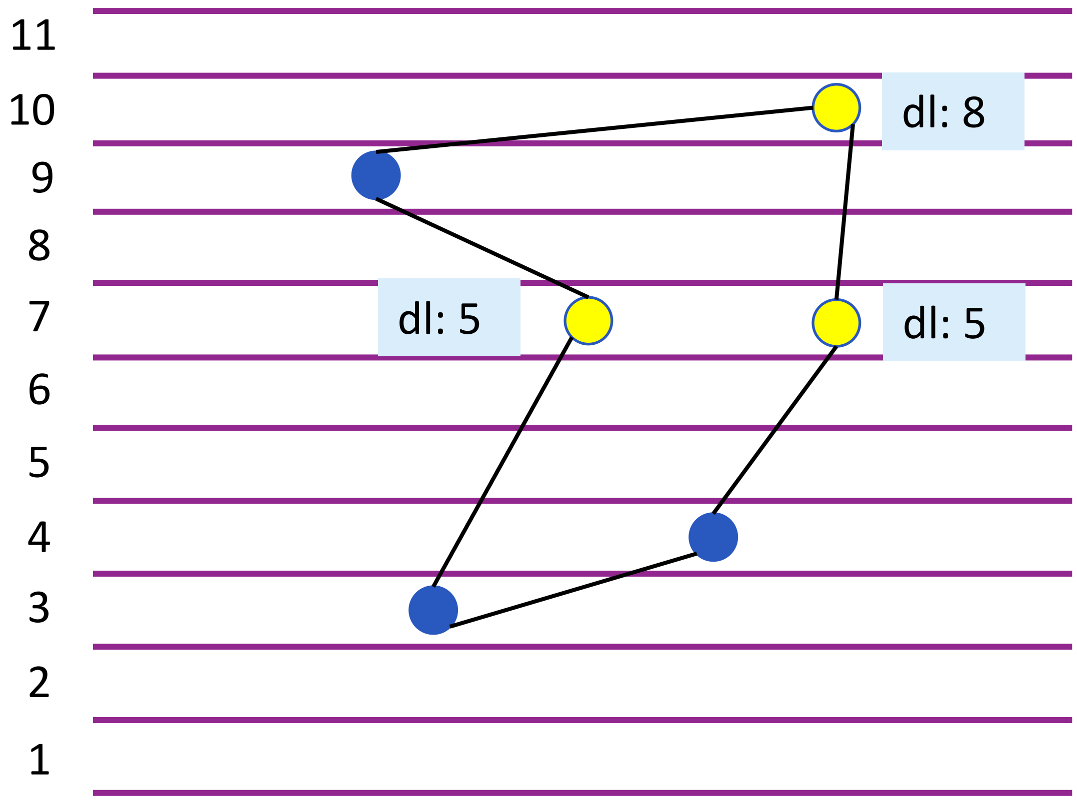

Iterate from **bottommost level to top level** and move vertices to desire-level

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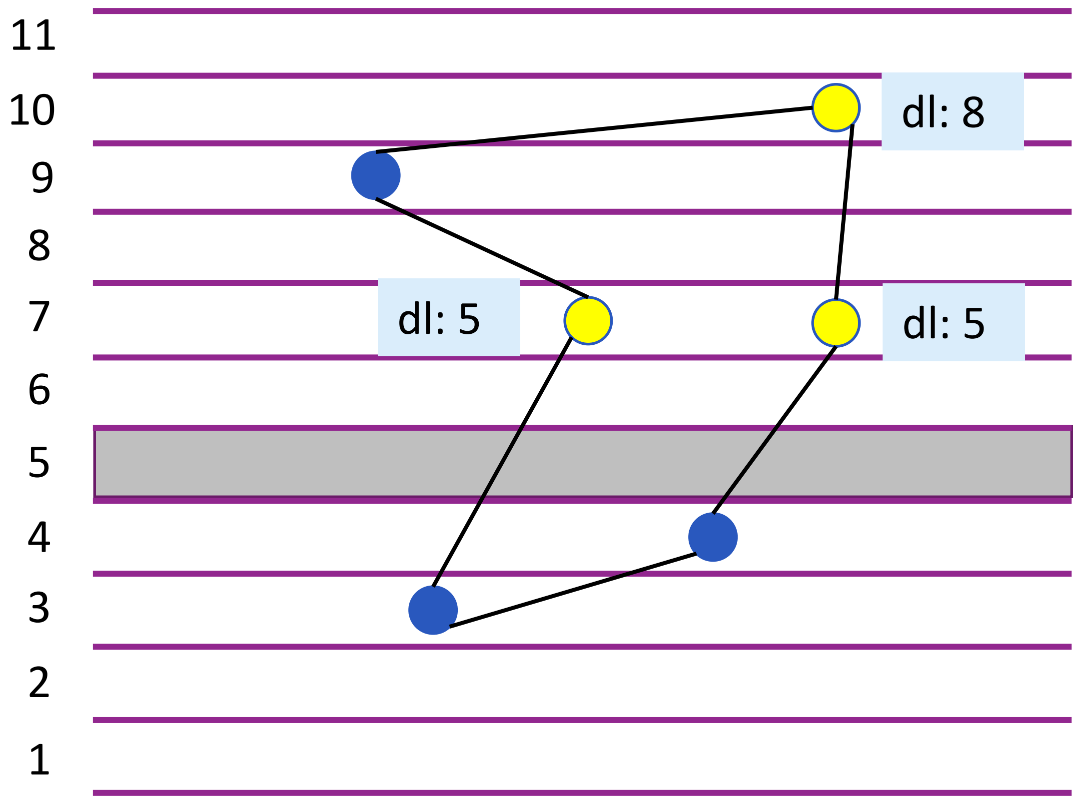

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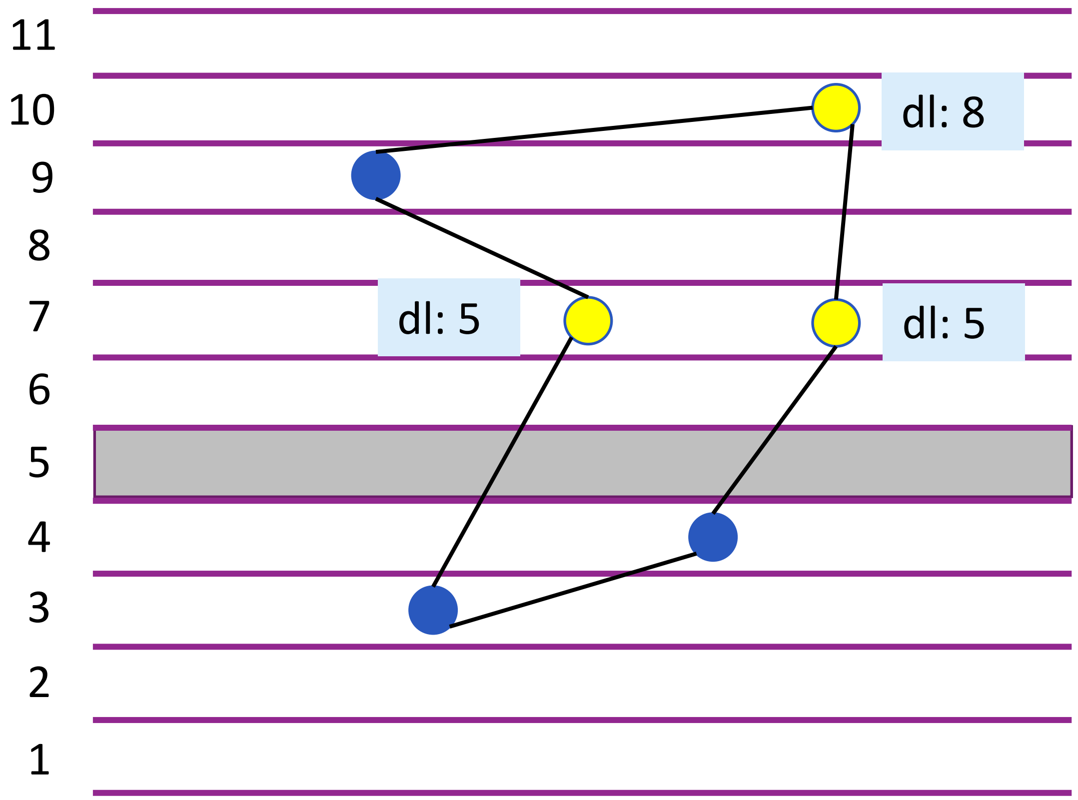

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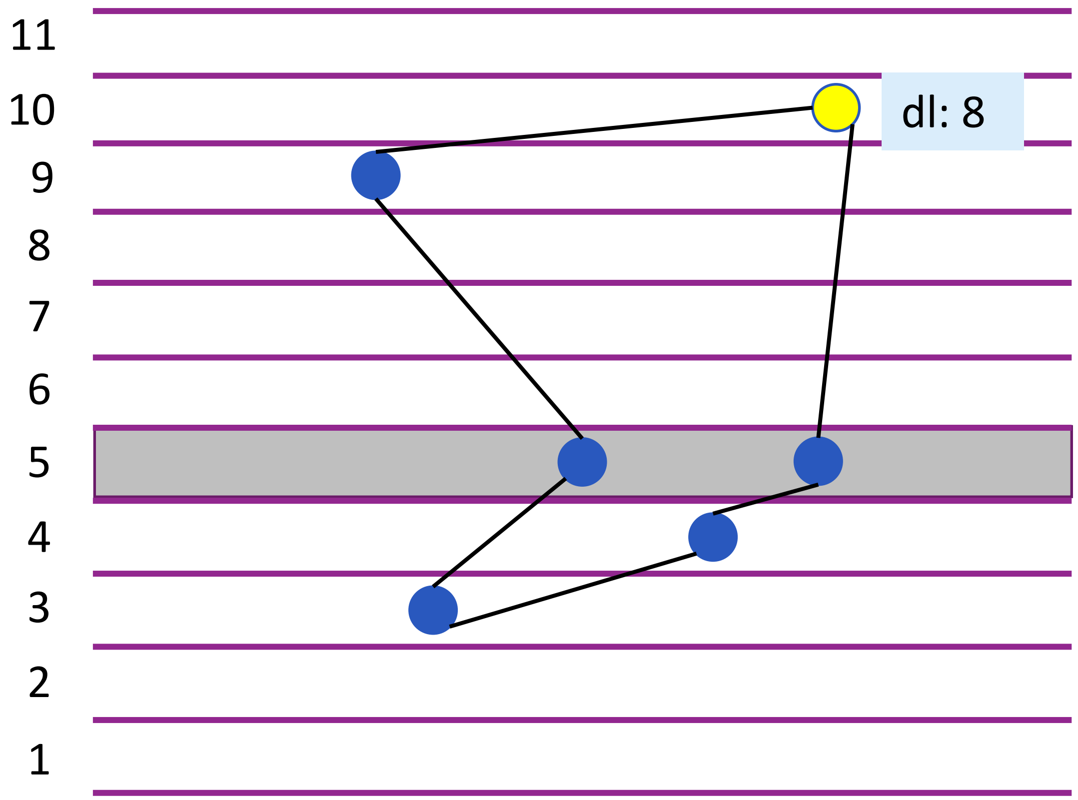

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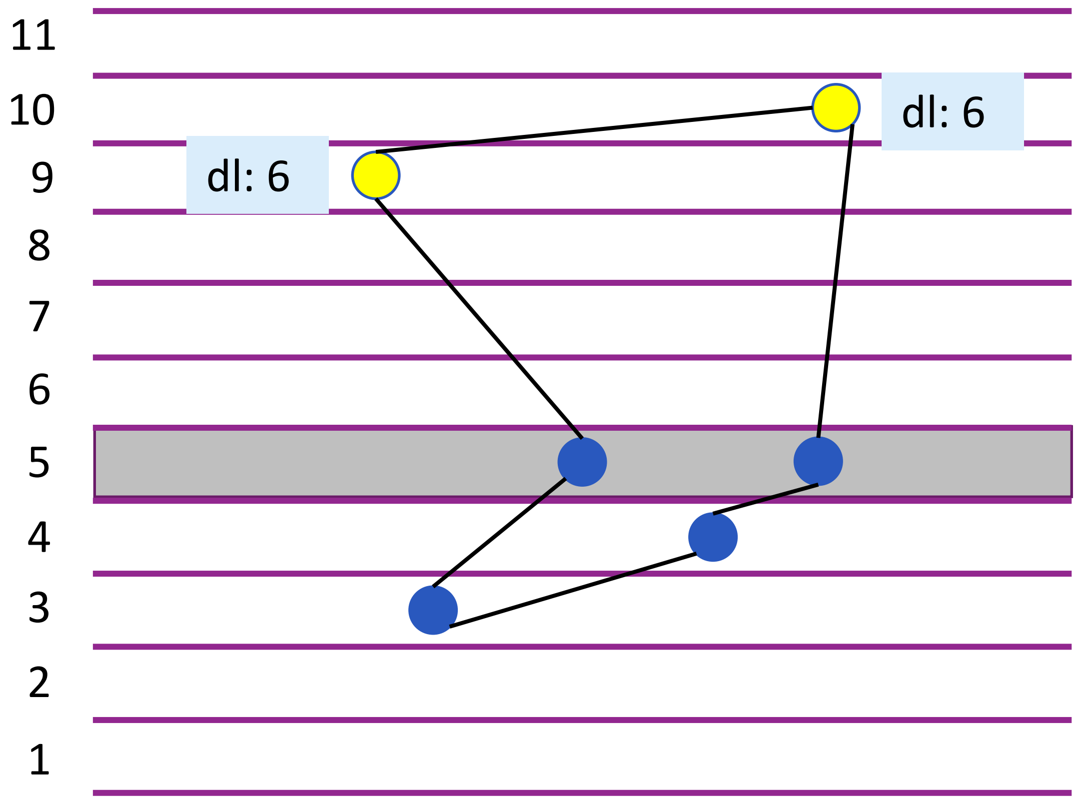

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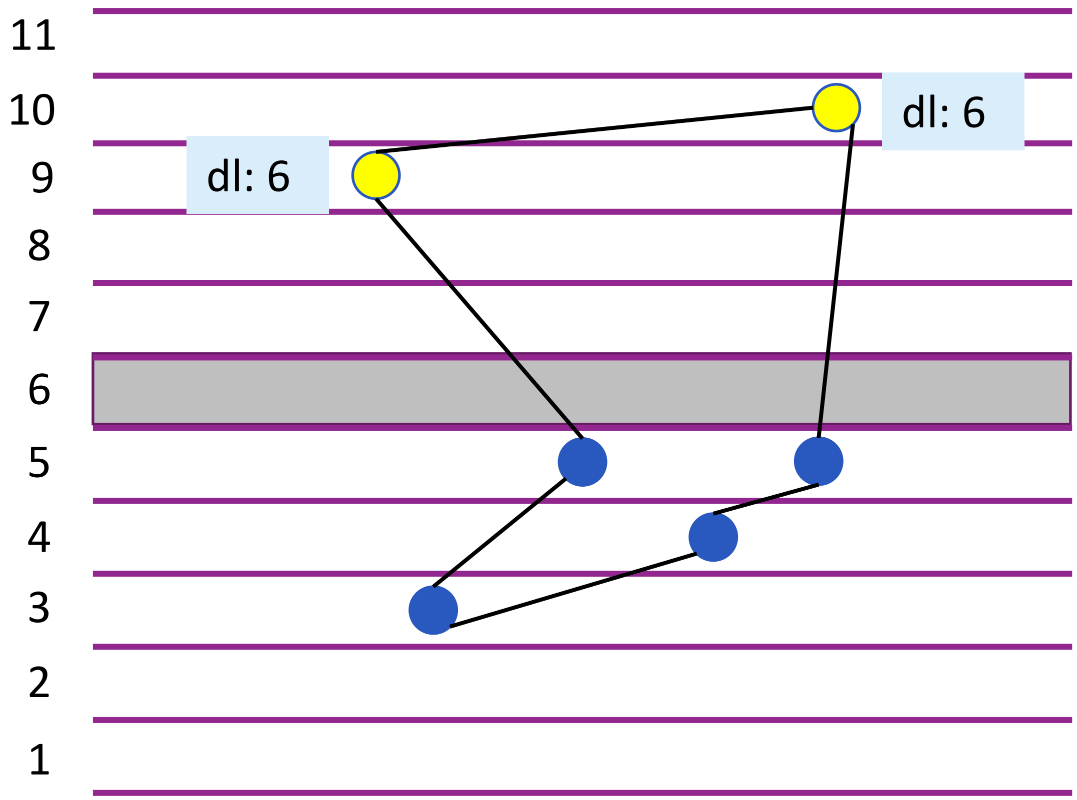

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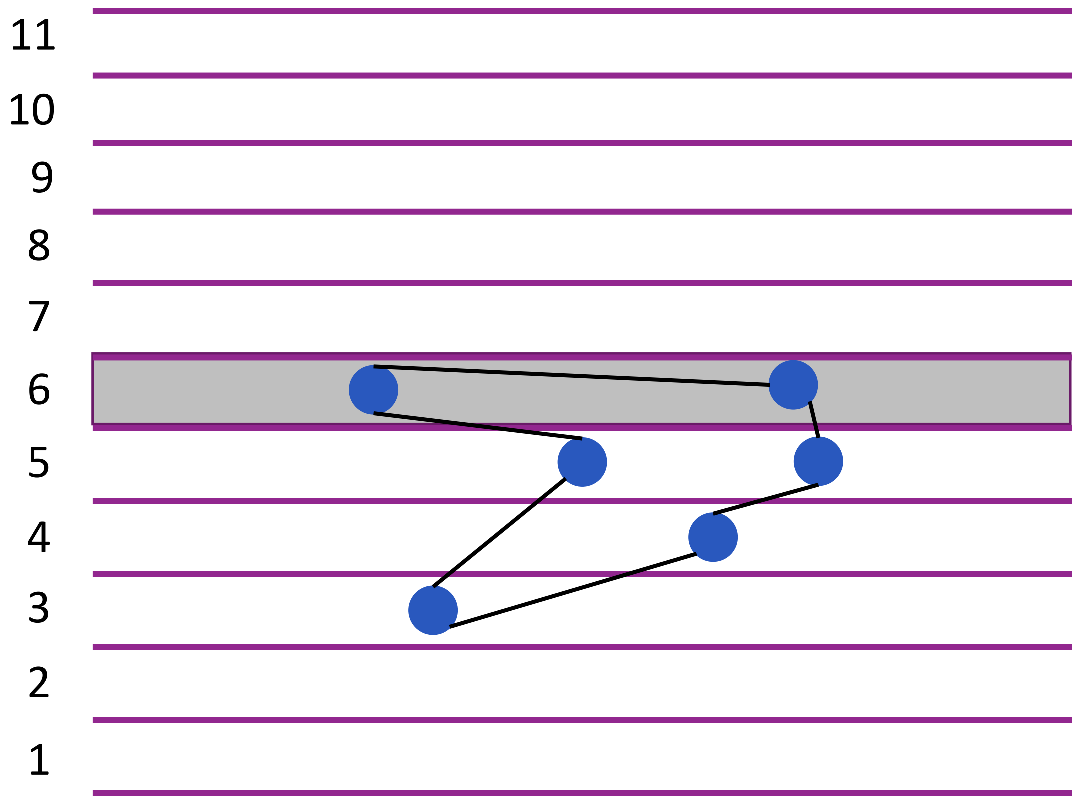
Calculate *desire-level*: **closest level that satisfies cutoffs**



Iterate from **bottommost level to top level** and move vertices to desire-level

Only lower bound cutoff, $(1+\epsilon)^i$, ever violated.

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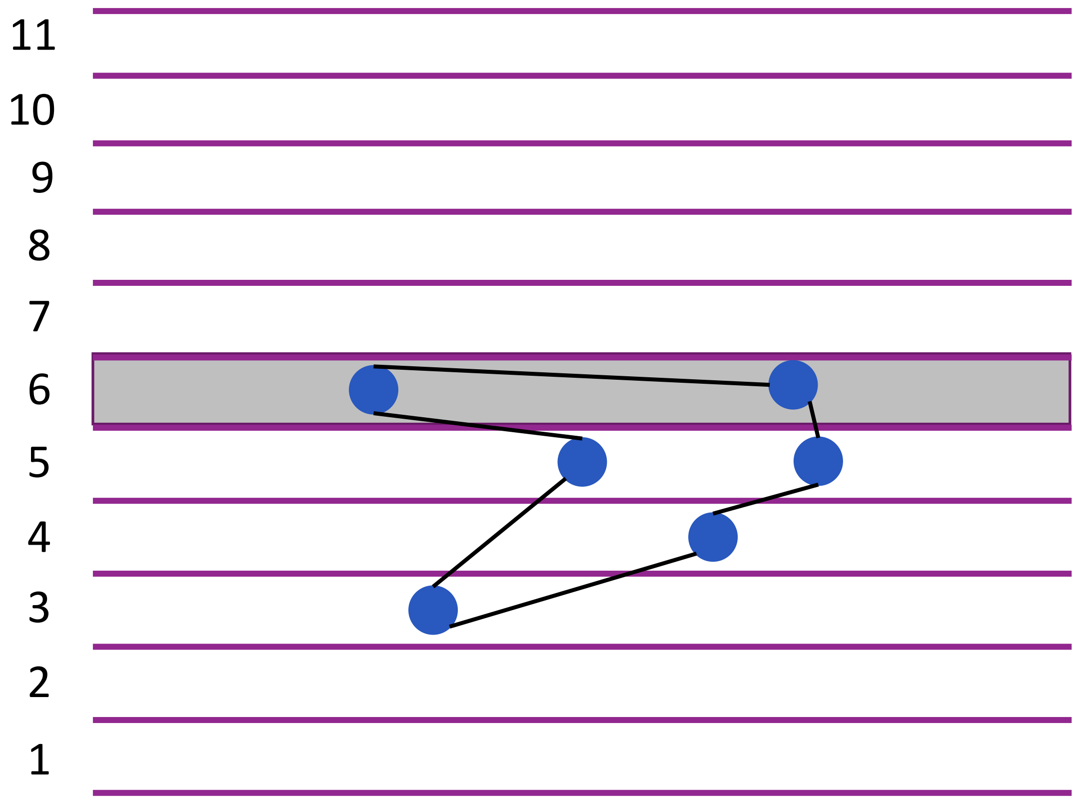

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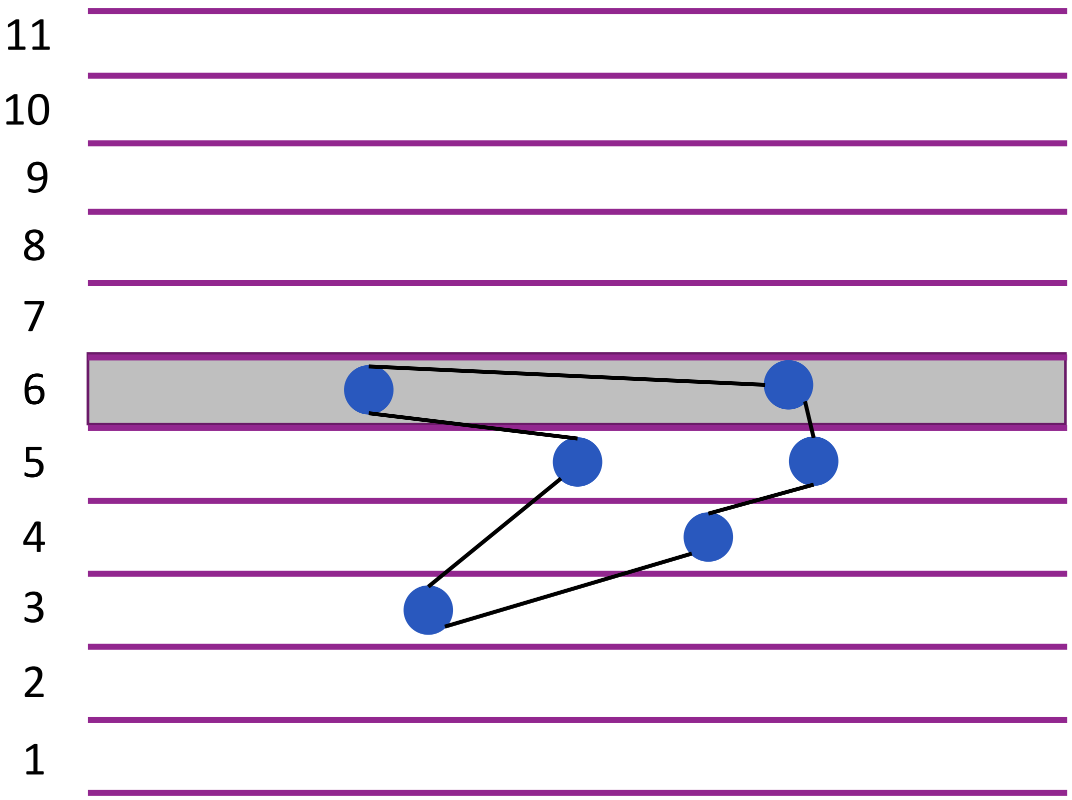

$O(\log^2 n \log\log n)$ depth w.h.p

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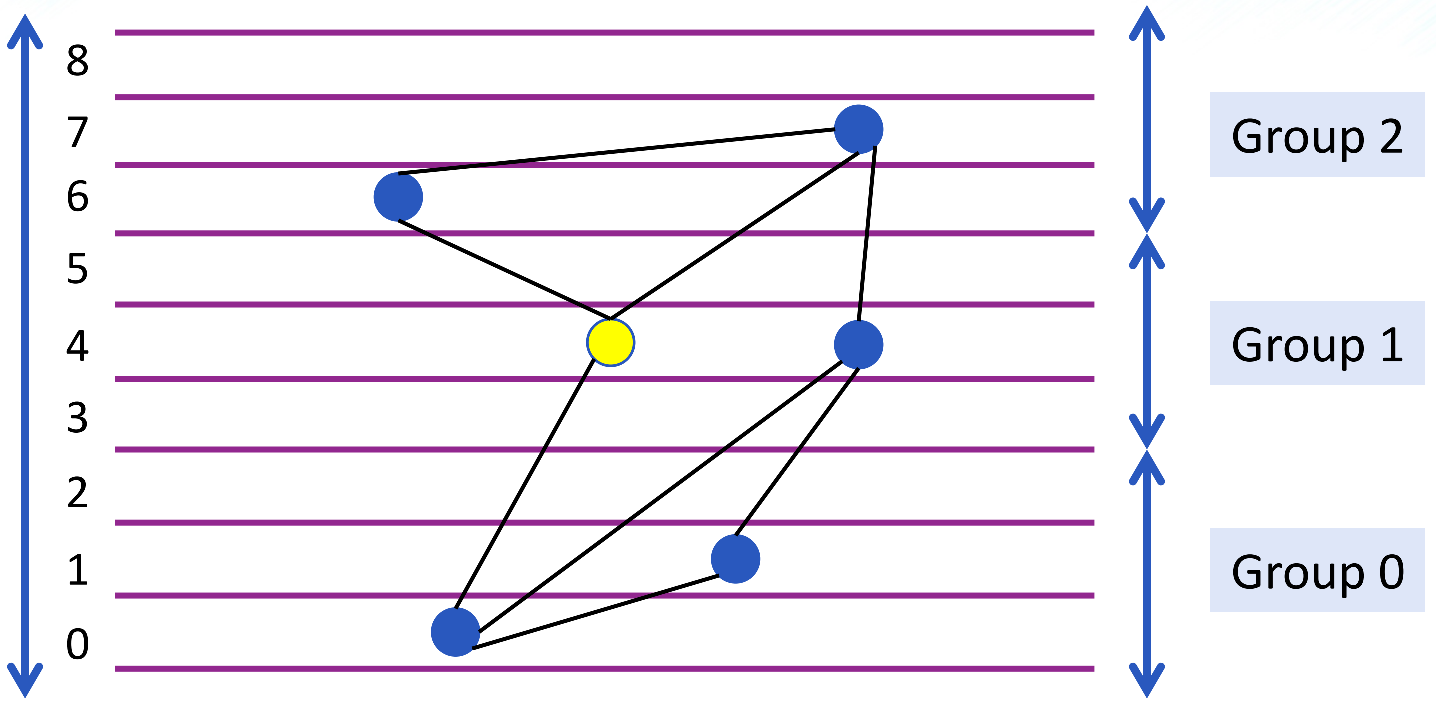
Obtaining the Coreness Estimate

$O(\log^2 n)$



- Set the coreness estimate: $\mathbf{(1+\delta)}^{\boldsymbol{max(\lfloor (level(v)+1)/\,(4\lceil \log_{1+\delta} n\rceil)\rfloor - 1, 0)}}$
- Each group has $4\lceil\log_{1+\delta} n\rceil$ levels

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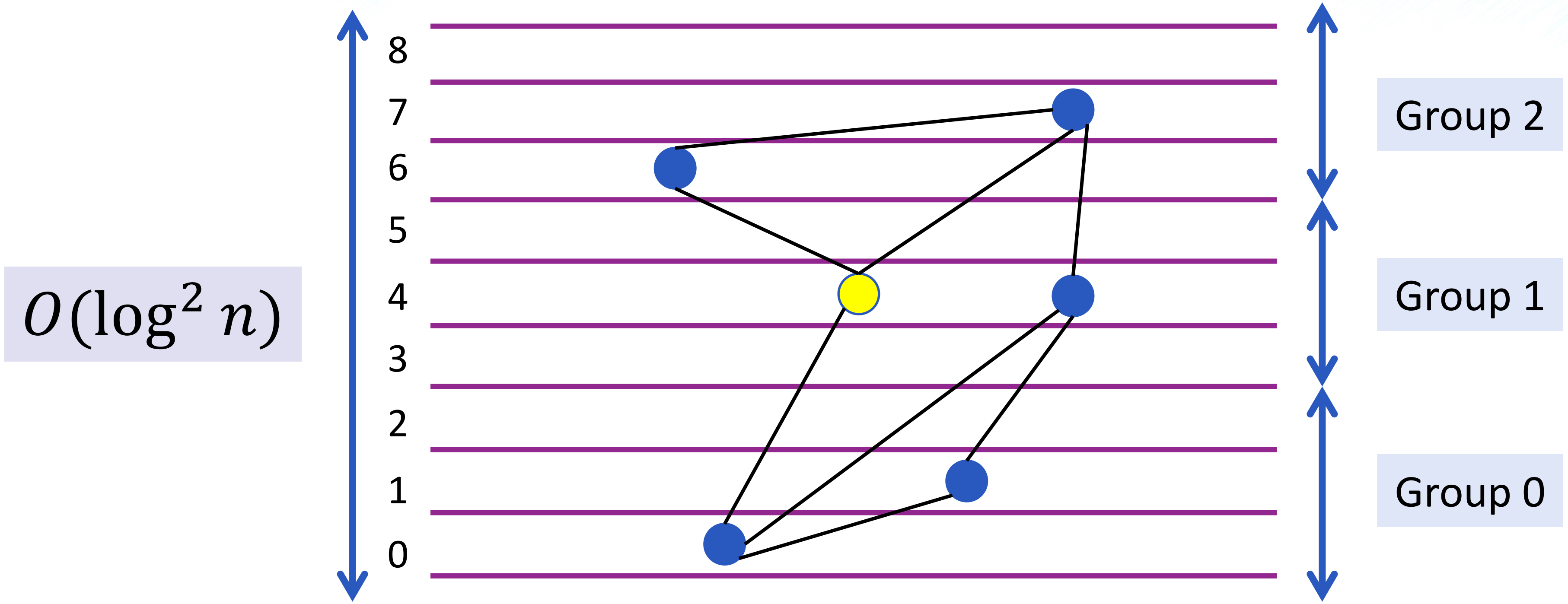

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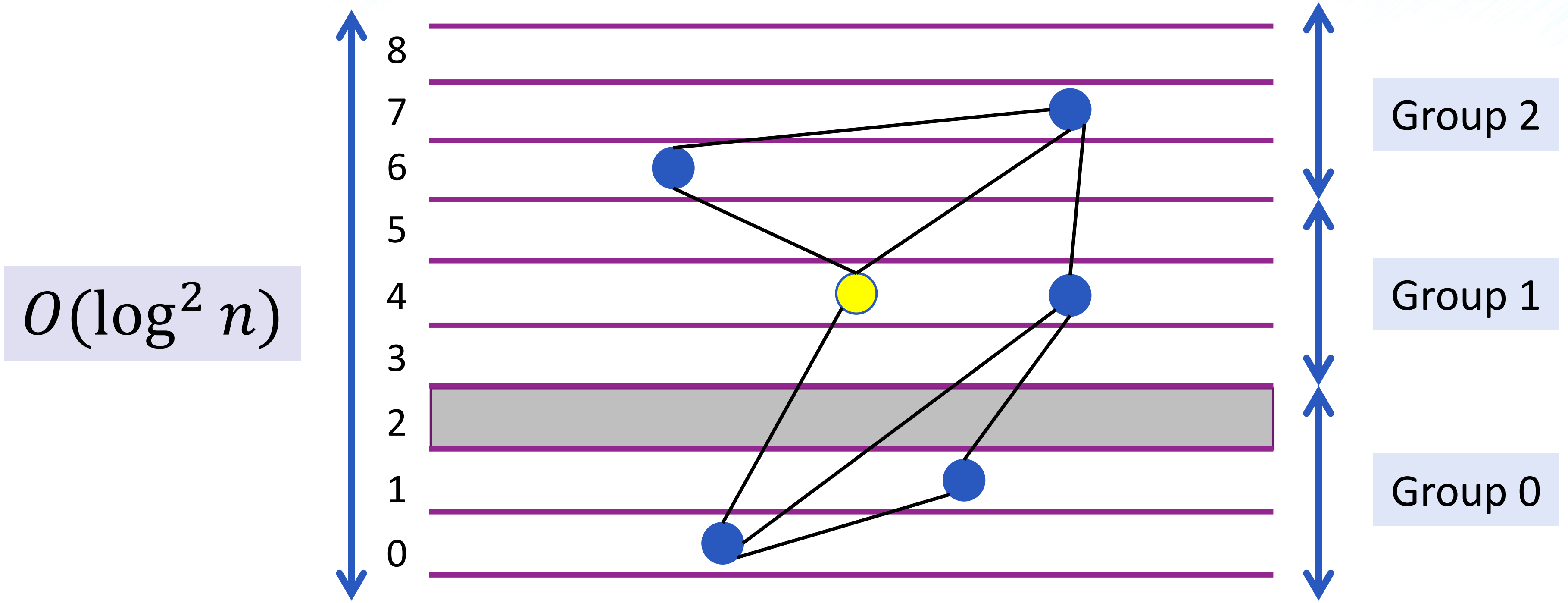

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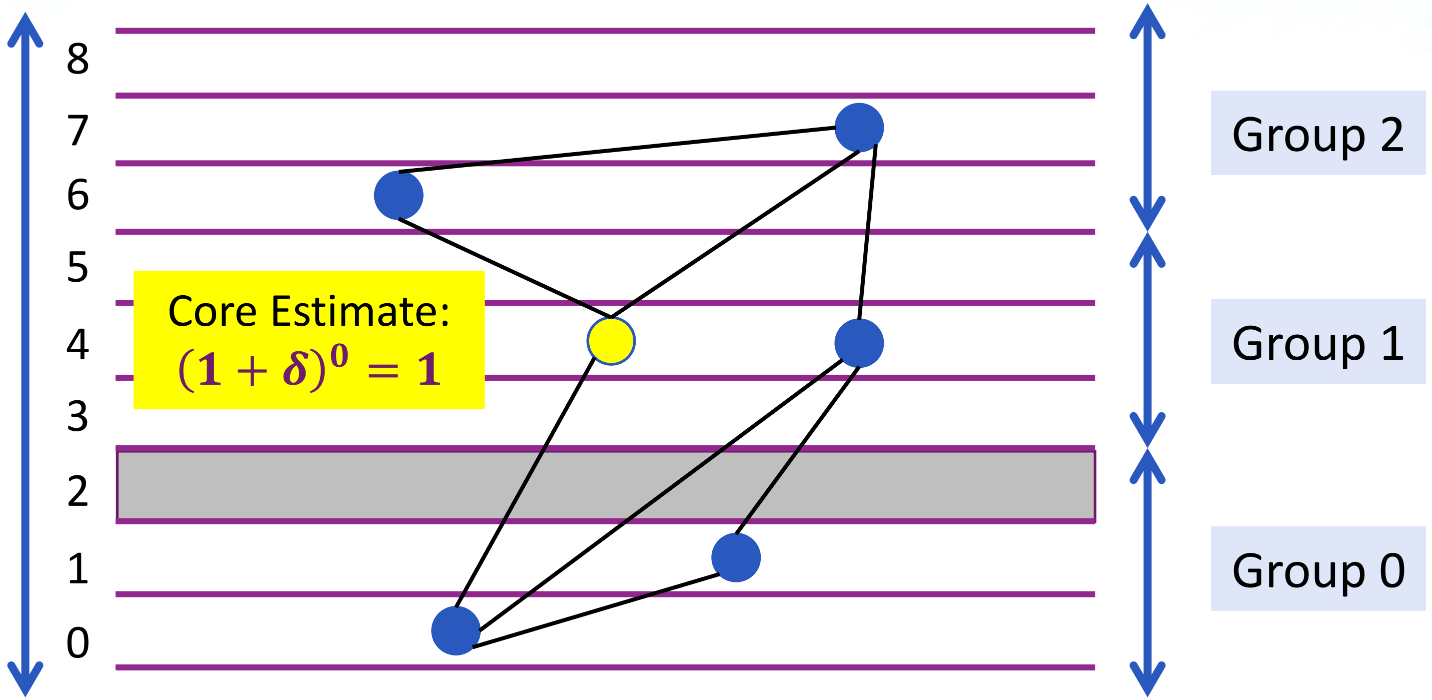

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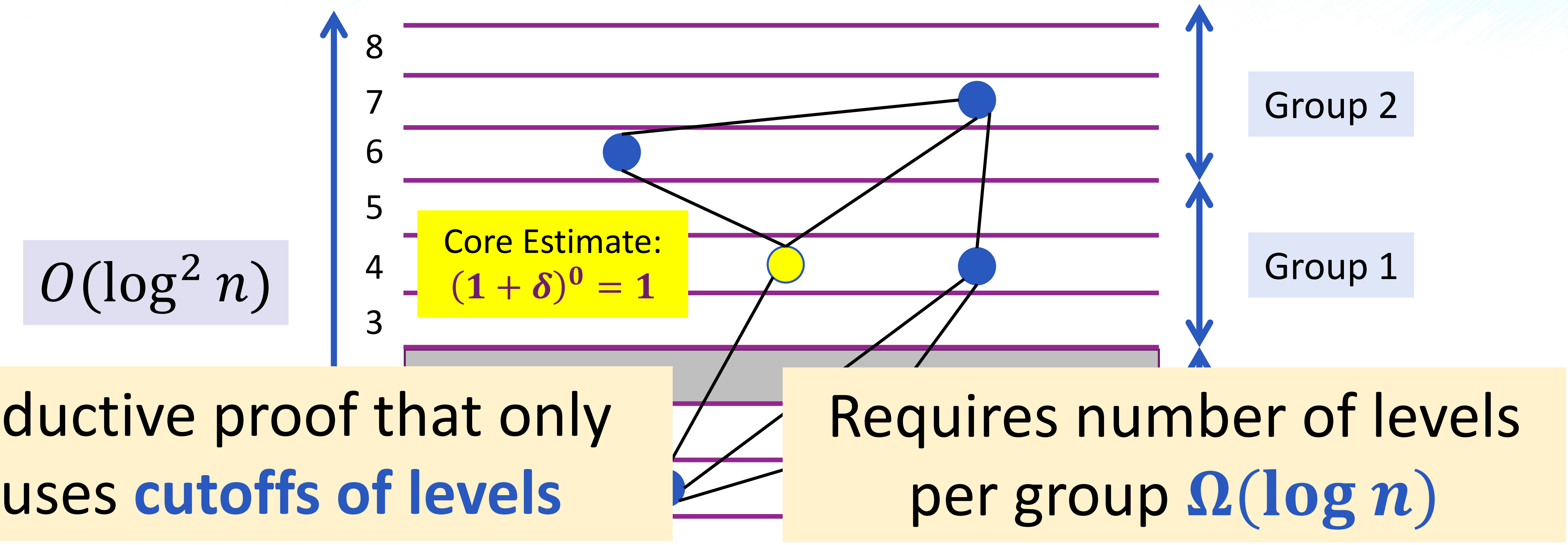

Inductive proof that only uses **cutoffs of levels**

Requires number of levels per group $\mathbf{\Omega(\log n)}$

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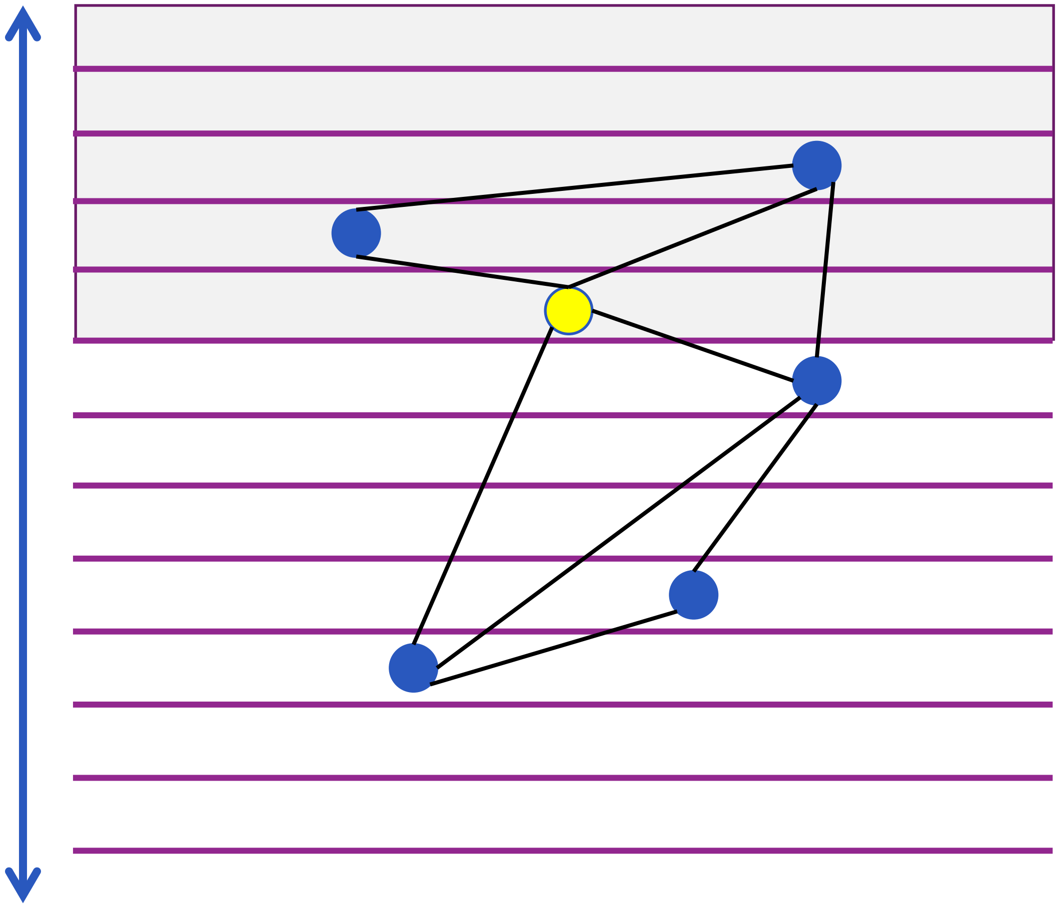
Proof of Our Approximation Factor: Upper Bound

Coreness Estimate:
$(1+\epsilon)^i$



Invariant: $\leq 2.1(1+\epsilon)^i$

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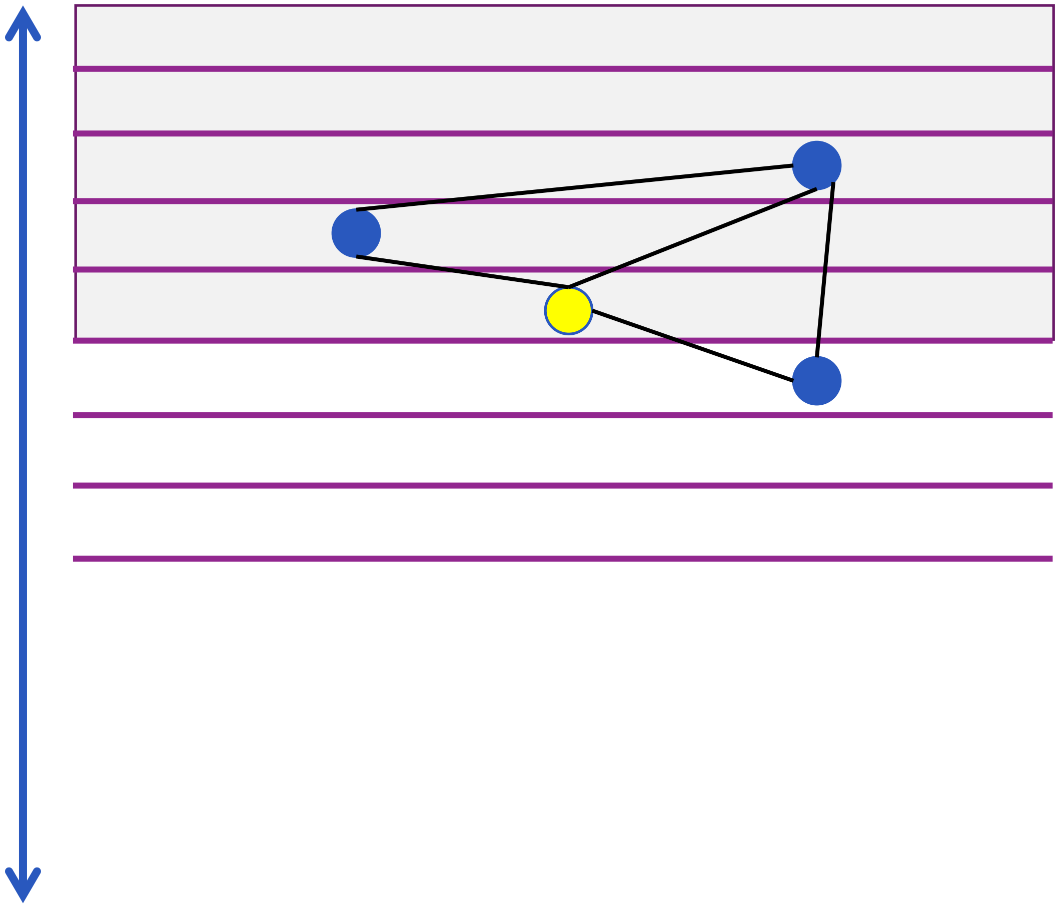

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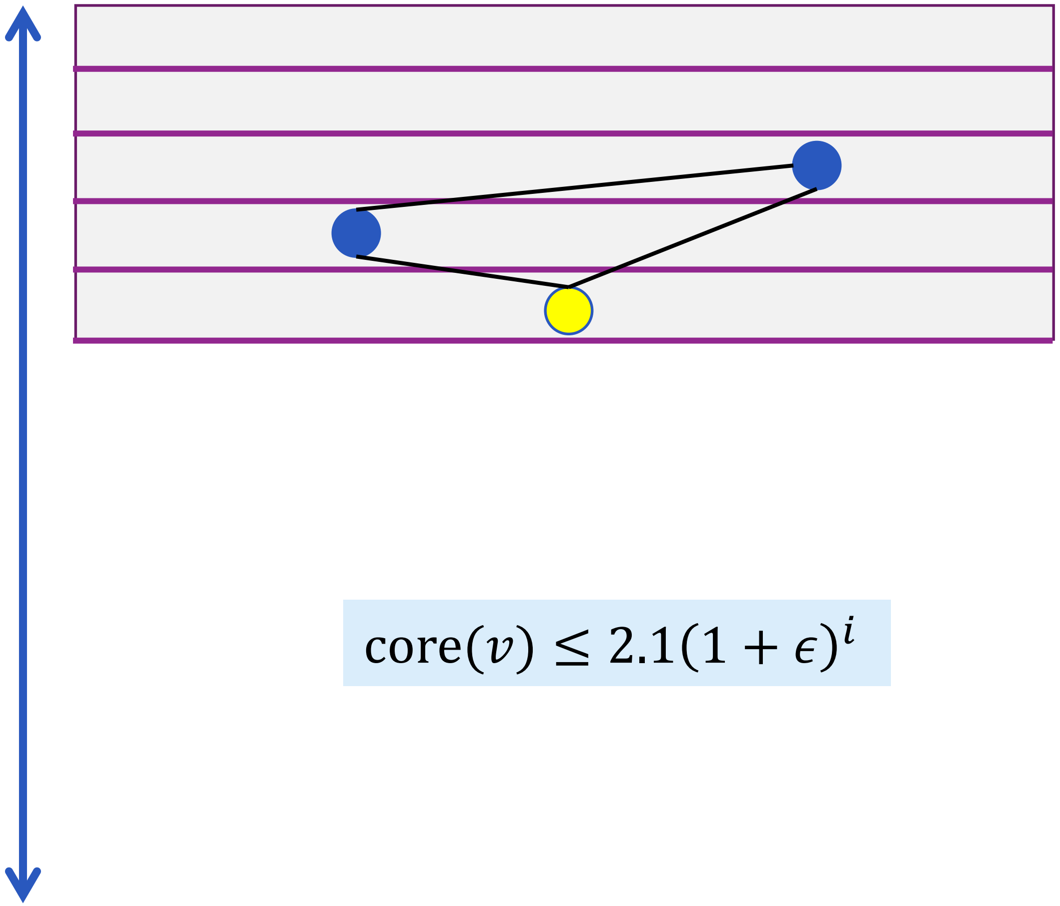

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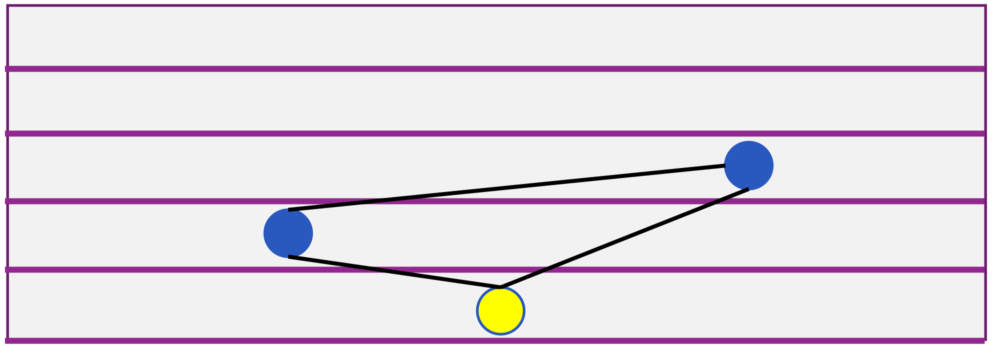

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Key Proof: Lower bound proof only requires lower bound invariant and definition of *k-core.*

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- **$O(B \log^2 n)$ amortized work** is based on potential argument
  - Vertices and edges store potential based on their levels

Experimental Implementation Details

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- Moving vertices around in the PLDS requires carefully updating these hash tables for work-efficiency

Tested Graphs

Graphs from Stanford SNAP database, DIMACS Shortest Paths challenge, and Network Repository—including some temporal

Graph	Num. Vertices	Num. Edges	Max *k*
dblp	425,957	2,099,732	101
brain-network	784,262	267,844,669	1200
wikipedia	1,140,149	2,787,967	124
youtube	1,138,499	5,980,886	51
stackoverflow	2,601,977	28,183,518	163
livejournal	4,847,571	85,702,474	329
orkut	3,072,627	234,370,166	253
usa-central	14,081,816	16,933,413	2
usa-road	23,072,627	28,854,312	3
twitter	41,652,231	1,202,513,046	2484
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Experiments

- c2-standard-60 Google Cloud instances
  - 30 cores with two-way hyper-threading
  - 236 GB memory
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Improvements across **all** experiments!

Runtimes/Accuracy Against State-of-the-Art Algorithms

Benchmarks

- **Sun et al. TKDD**: **sequential, approx., dynamic** algorithm
- **LDS**: **sequential, approx., dynamic** LDS of Henzinger et al.
- **Zhang and Yu SIGMOD**: **sequential, exact, dynamic** algorithm
- **Hua et al. TPDS: parallel, exact, dynamic** algorithm

Versions of PLDS

- **PLDS: exact theoretical** algorithm
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Key Optimization Feature:
Reduce number of levels per group

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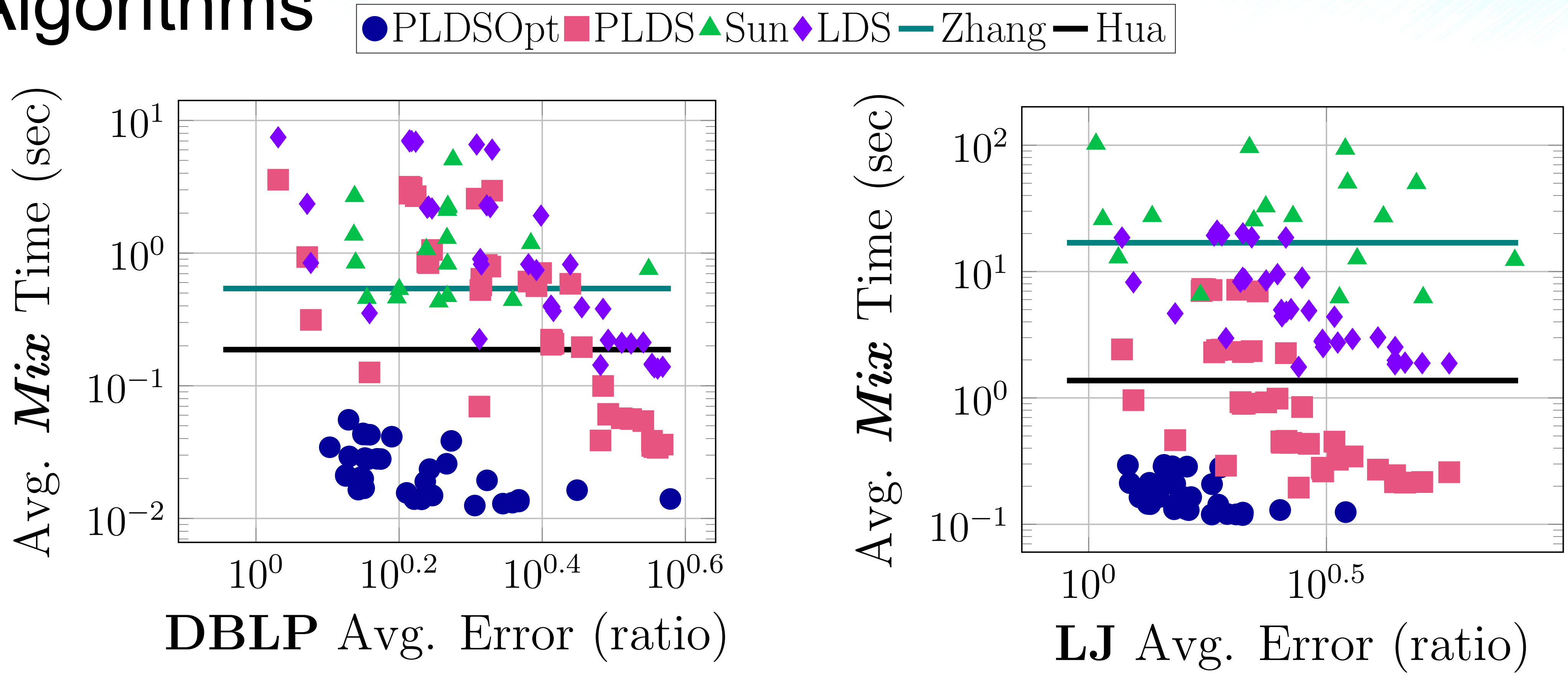
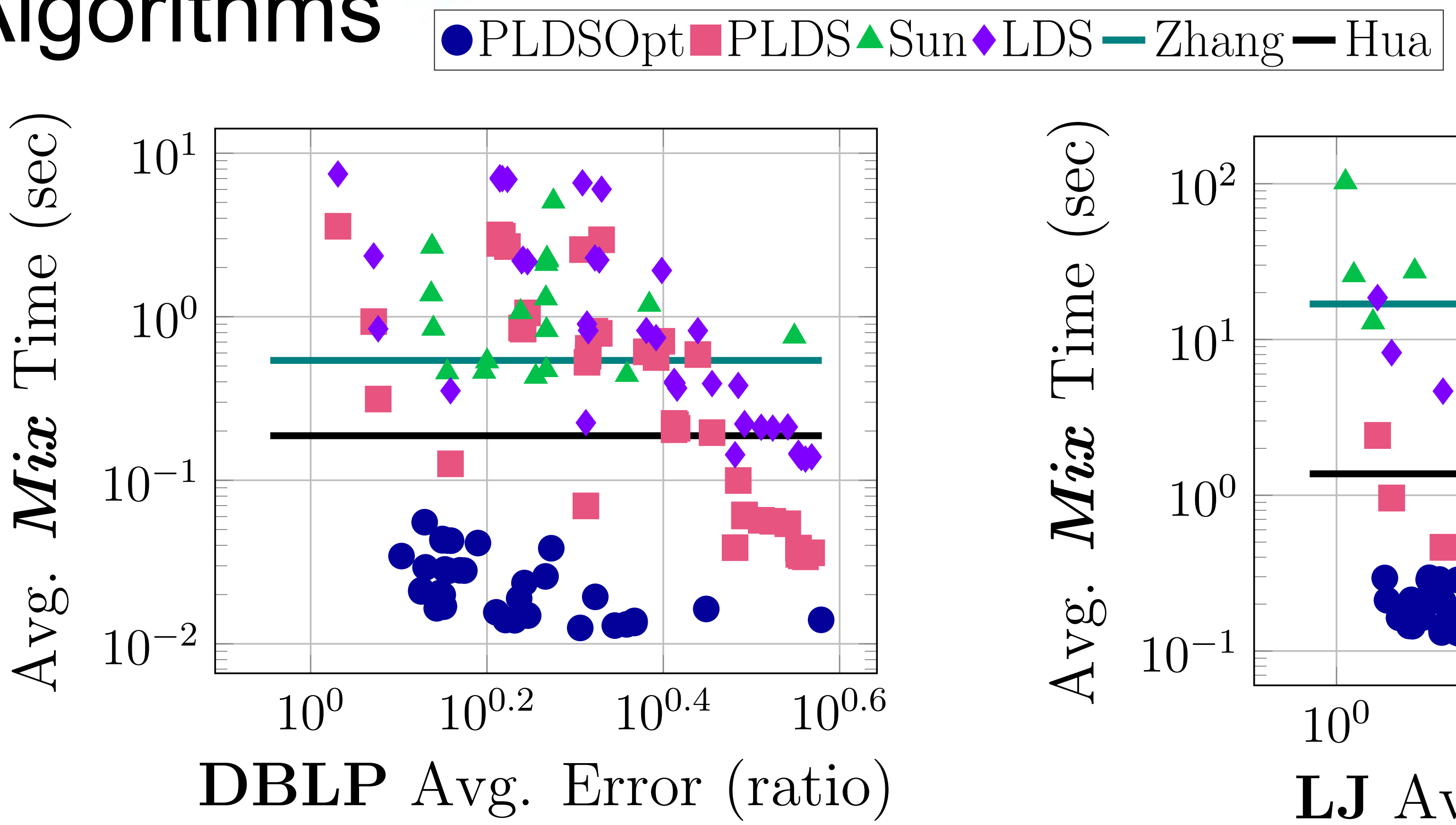
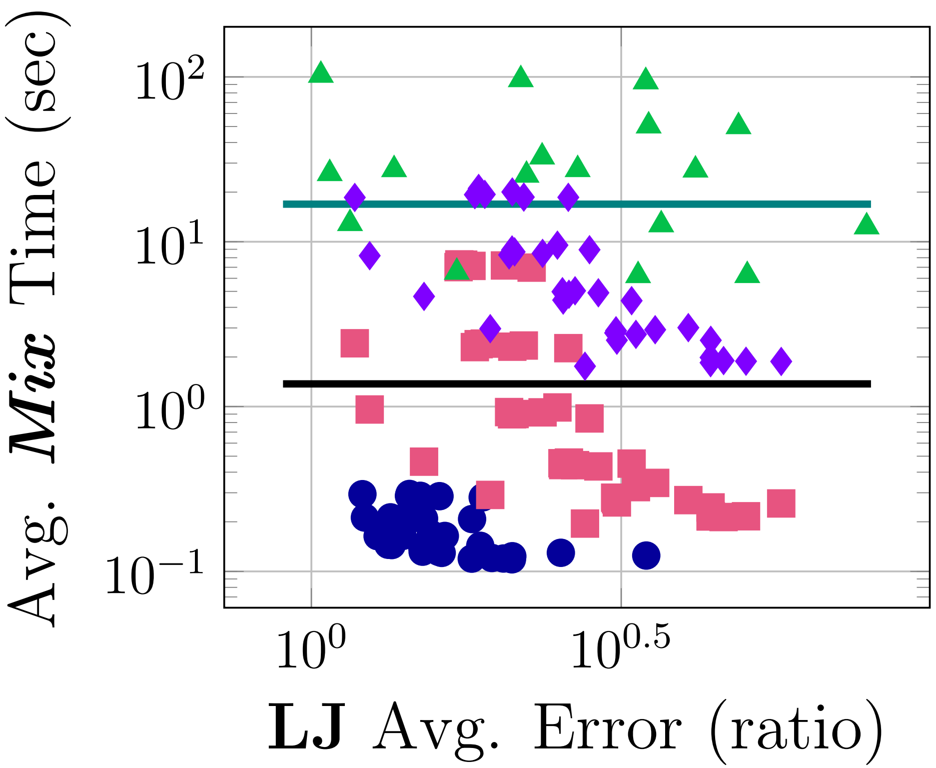


DBLP: 425K vertices, 2.1M edges

LJ (LiveJournal): 4.8M vertices, 85.7M edges

Runtimes/Accuracy Against State-of-the-Art Algorithms





PLDSOpt: 19.04–544.22x
speedup over Sun

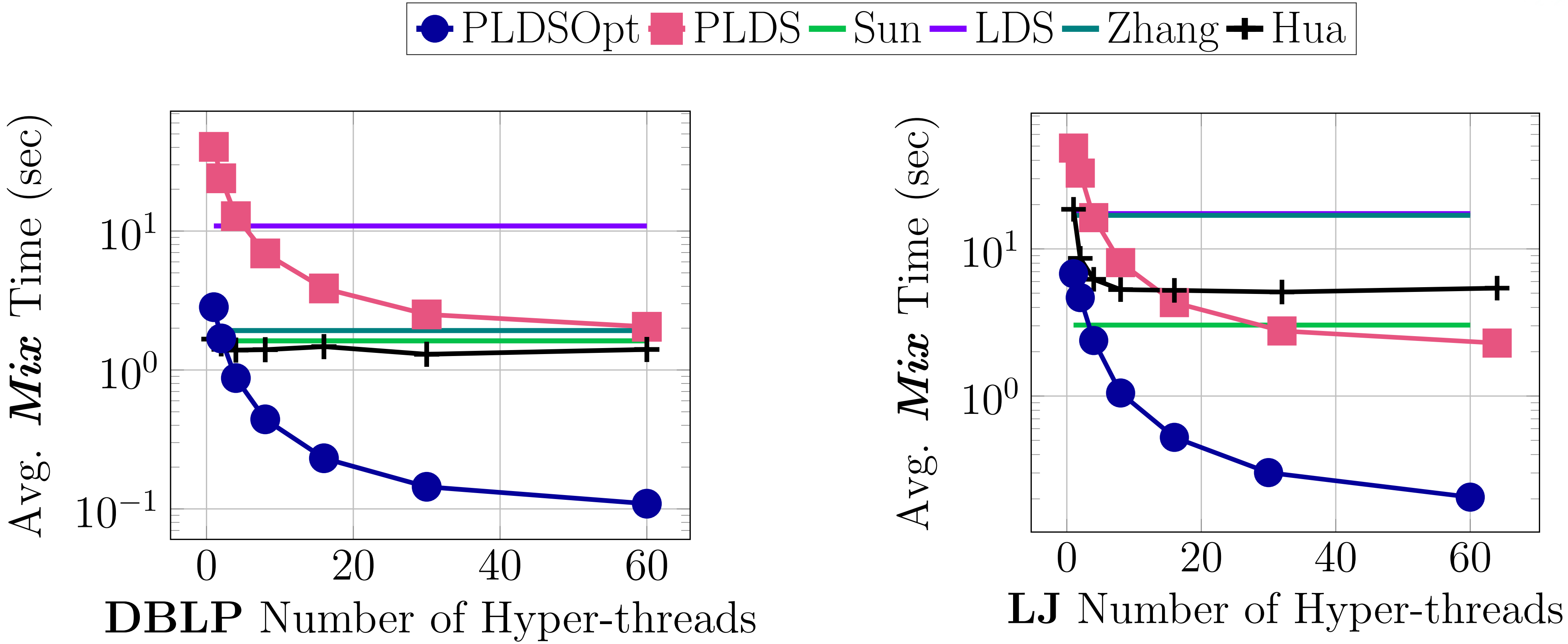
LJ (LiveJ

PLDSOpt: 2.49–24.41x
speedup over Hua

edges

Number of Hyper-Threads

Faster than all other algorithms at 4 cores!



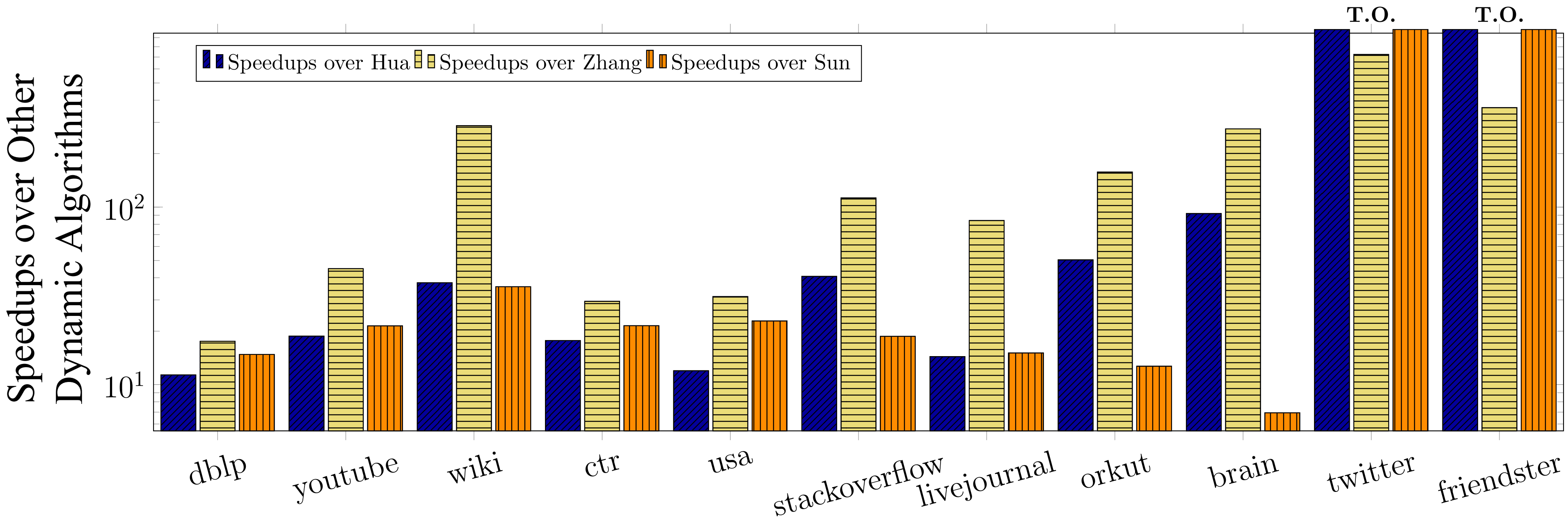
PLDSOpt: 33.02x
self-relative speedup

PLDS: 26.46x
self-relative speedup

Hua: 3.6x
self-relative speedup

Speedups On a Variety of Graphs

- Speedups against dynamic benchmarks: Hua, Zhang, and Sun



Batch size = 10^6

Graphs ordered by size (left to right)

Speedups on **all graphs** against **all benchmarks**

Speedups up to: **91.95x** for Hua, **35.59x** for Sun, **723.72x** for Zhang

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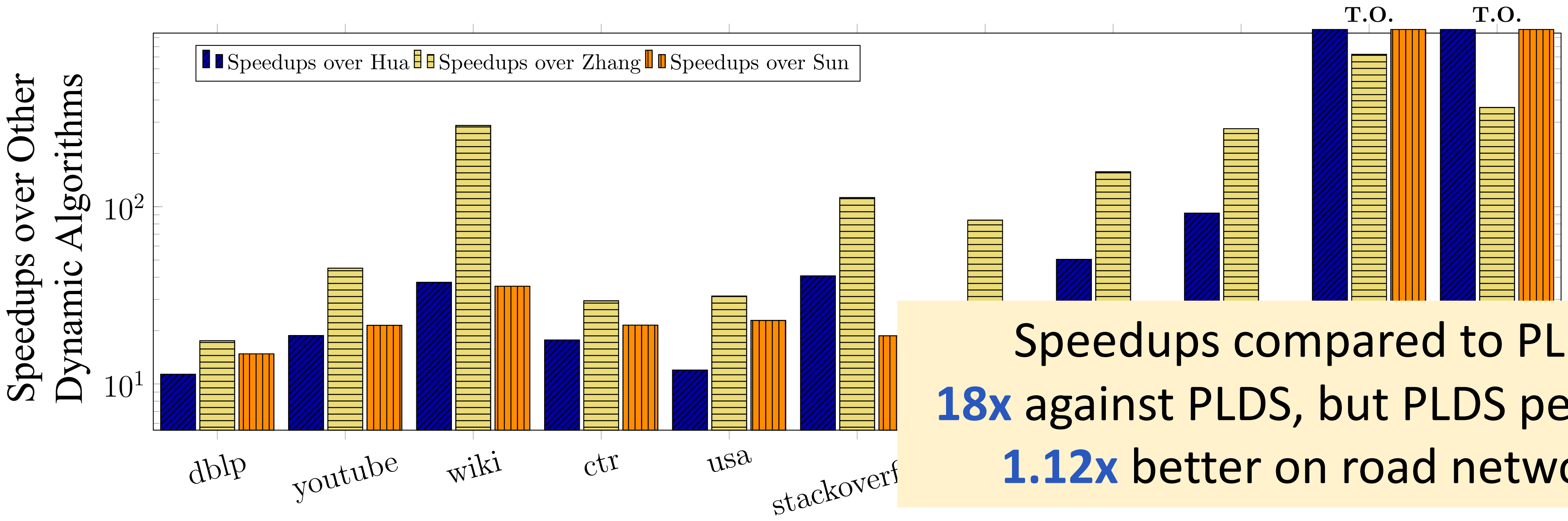

Speedups compared to PLDS:
18x against PLDS, but PLDS performs **1.12x** better on road networks

Batch size = 10^6

Graphs ordered by size (left to right)

Speedups on **all graphs** against **all benchmarks**

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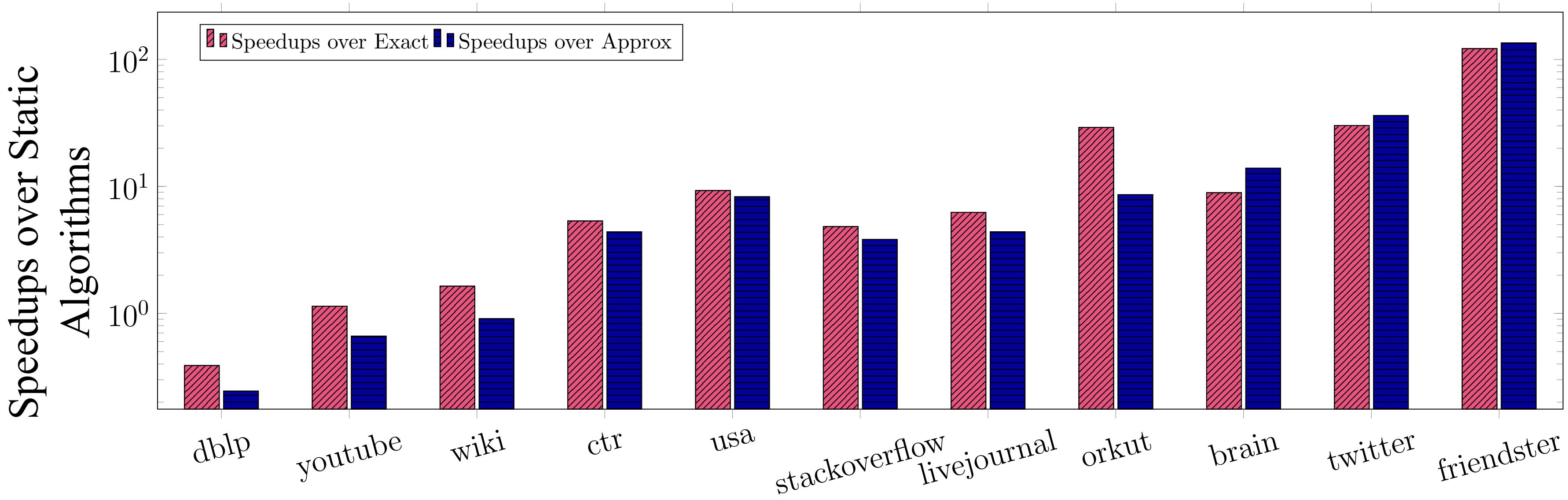
Speedups Against Parallel Static Algorithms

- Parallel exact *k*-core decomposition [Dhulipala et al. ‘18]
- Parallel $(2+\epsilon)$-approximate *k*-core decomposition



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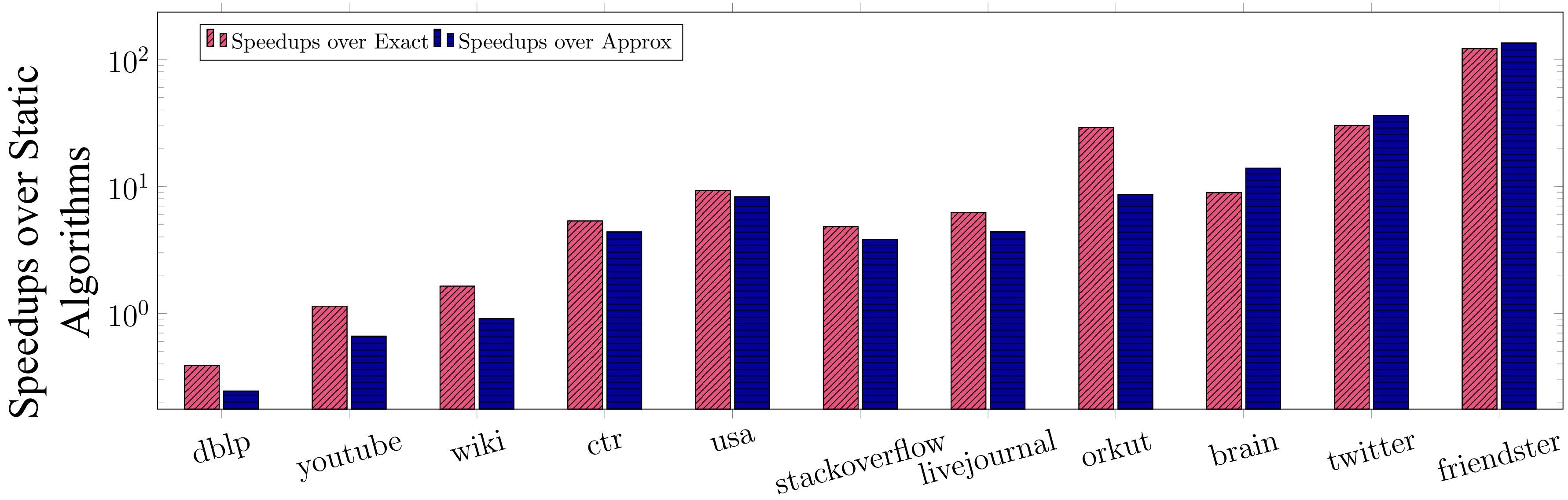


Batch size = 10^6

Graphs ordered by size (left to right)

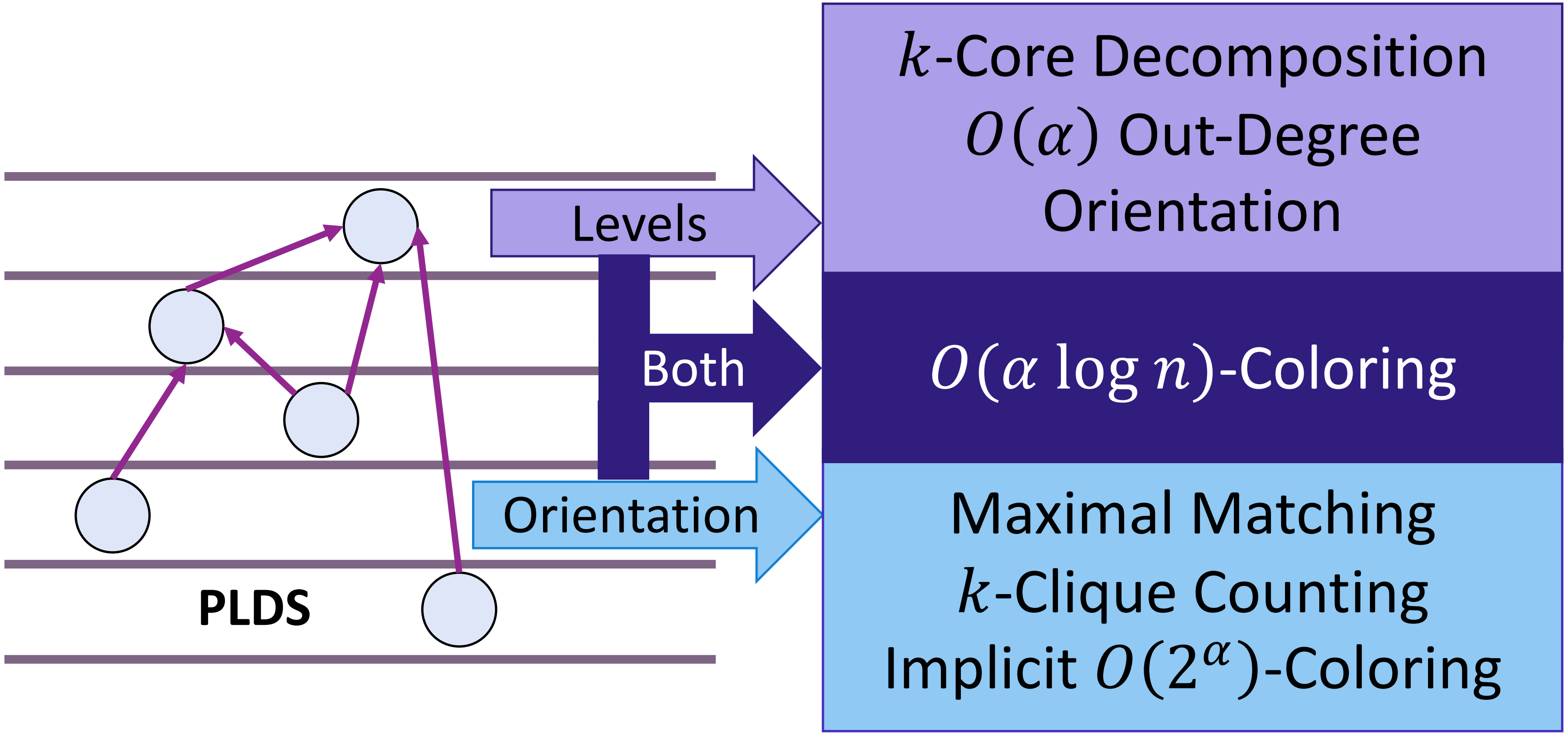
We achieve speedups for all but the smallest graphs

Speedups of up to **122x** for Twitter (1.2B edges) and Friendster (1.8B edges)

Other Results

Problem	Approx	Work	Depth
Static k-Core	$2+\varepsilon$	$O(m+n)$	$\tilde{O}(\log^2 n)$
Low Out-Degree	$4+\varepsilon$	$O(\lvert B\rvert \log^2 n)$	$\tilde{O}(\log^3 n)$
Maximal Matching	Maximal	$O(\lvert B\rvert\,(k+\log^2 n))$	$\tilde{O}(\log\Delta \log^2 n)$
Clique Counting	Exact	$O(\lvert B\rvert\,(k^{c-2}+\log^2 n))$	$\tilde{O}(\log^2 n)$
Explicit Coloring	$O(k \log n)$	$O(\lvert B\rvert \log^2 n)$	$\tilde{O}(\log^2 n)$
Implicit Coloring	$O(2^k)$	$O(\lvert B\rvert \log^3 n)$	$\tilde{O}(\log^2 n)$

PLDS to Other Results



Other Results + Future Work Implementations!

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Conclusion

- New parallel level data structure (PLDS)
- Parallel batch-dynamic algorithms for *k*-core decomposition and related problems (low out-degree orientation, maximal matching, clique counting, graph coloring)
- Our *k*-core algorithm achieves significant improvements over state-of-the-art solutions in practice
- Source code available at https://github.com/qqliu/batch-dynamic-kcore-decomposition

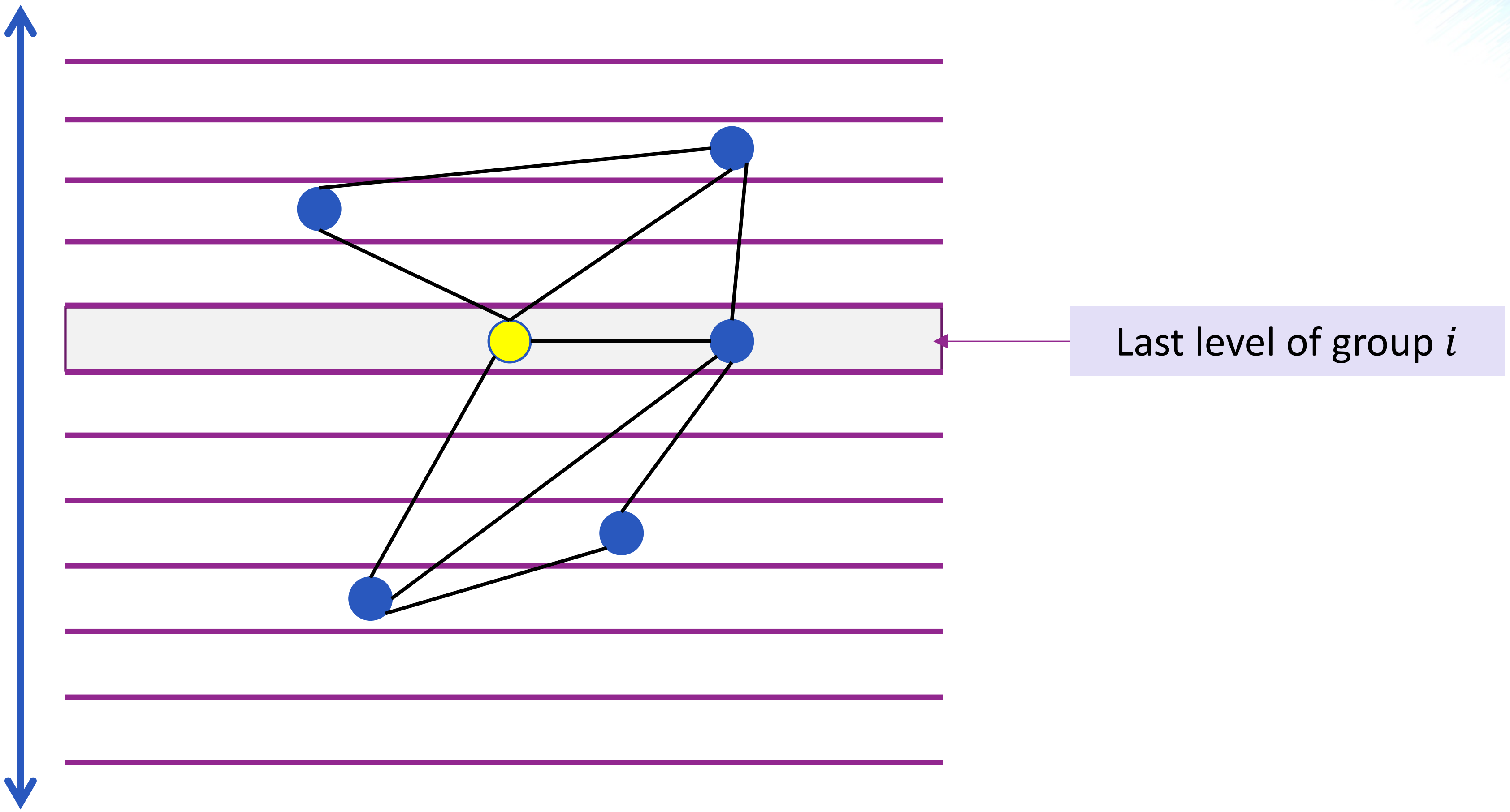
Extra Slides

Proof of Our Approximation Factor: Lower Bound

Estimate: $(1+\epsilon)^i$

Assume for Contradiction:

$$c(v) < \frac{(1+\epsilon)^i}{2.1(1+\epsilon)}$$



Proof of Our Approximation Factor: Lower Bound

Estimate: $(1+\epsilon)^i$

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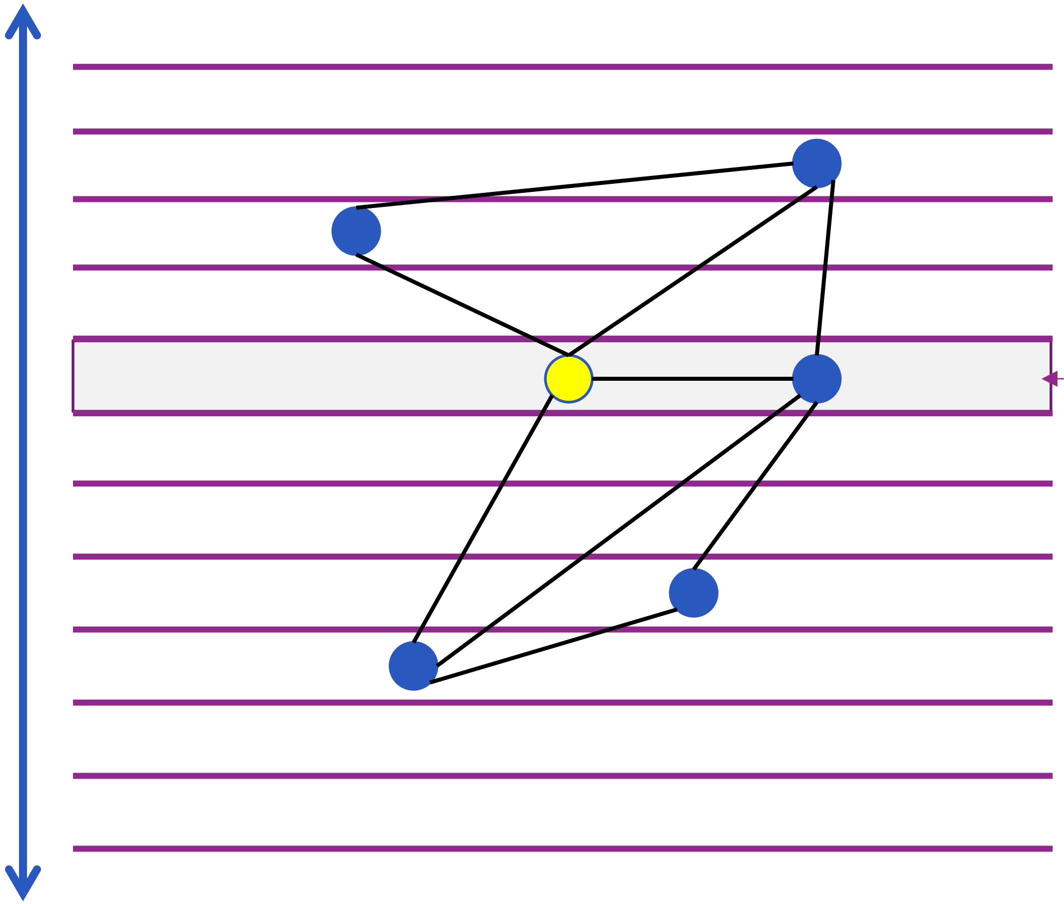

Last level of group i

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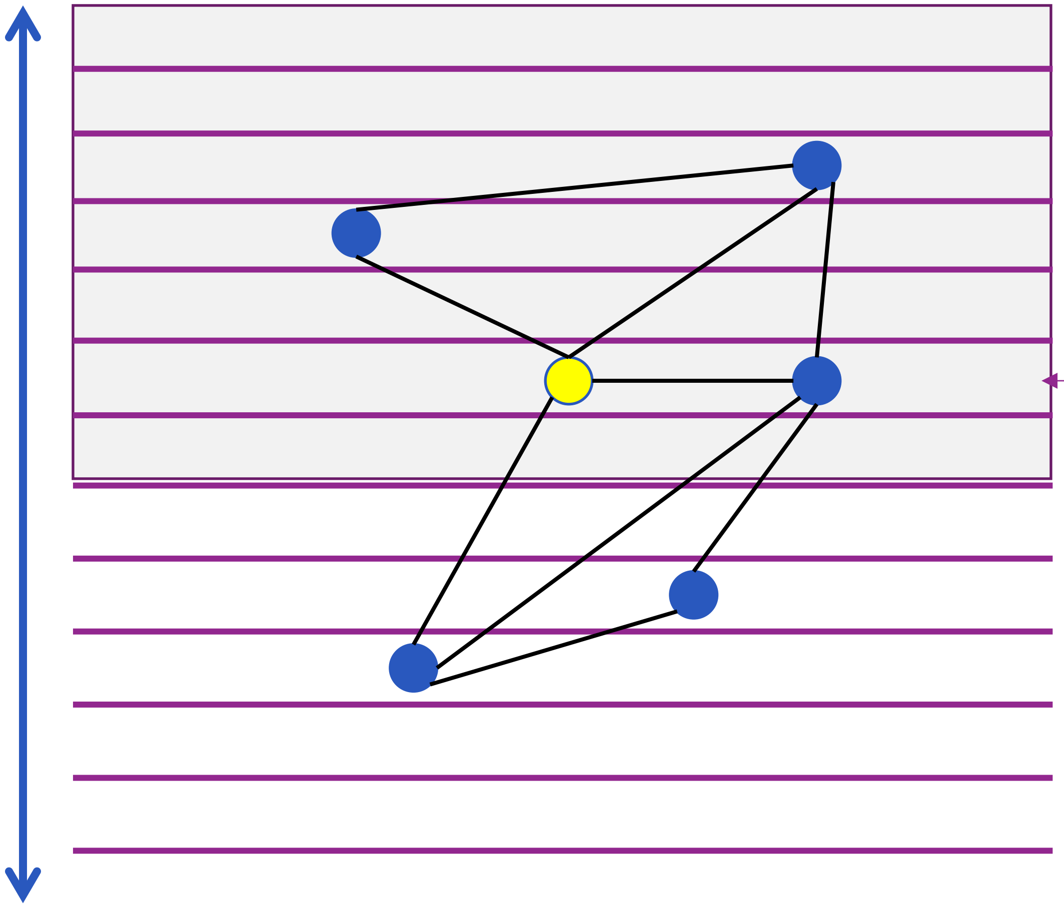

Last level of group i

nodes at or above level below v is: $\geq (1+\epsilon)^i$

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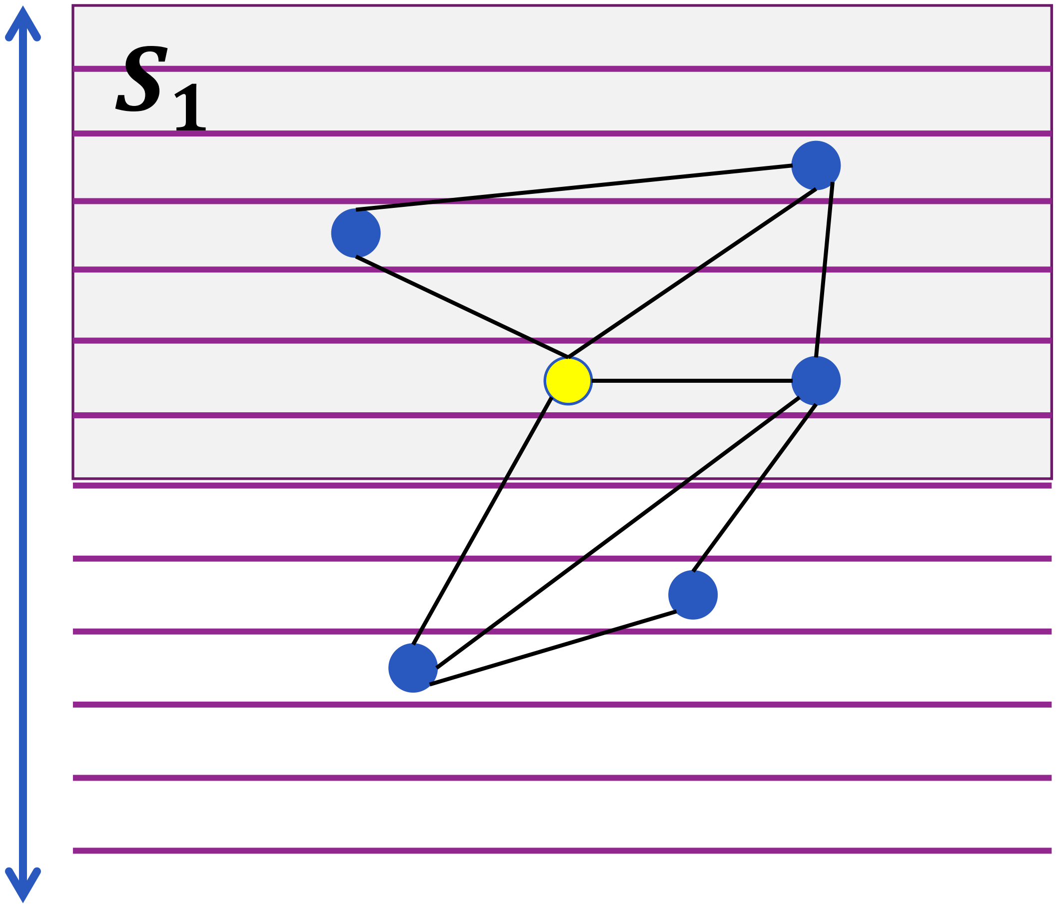


Pruning Procedure

Remove all w where

$$d_{S_1}(w) < \frac{(1+\epsilon)^i}{2.5}$$

nodes at or above level of v is: $\geq (1+\epsilon)^i$

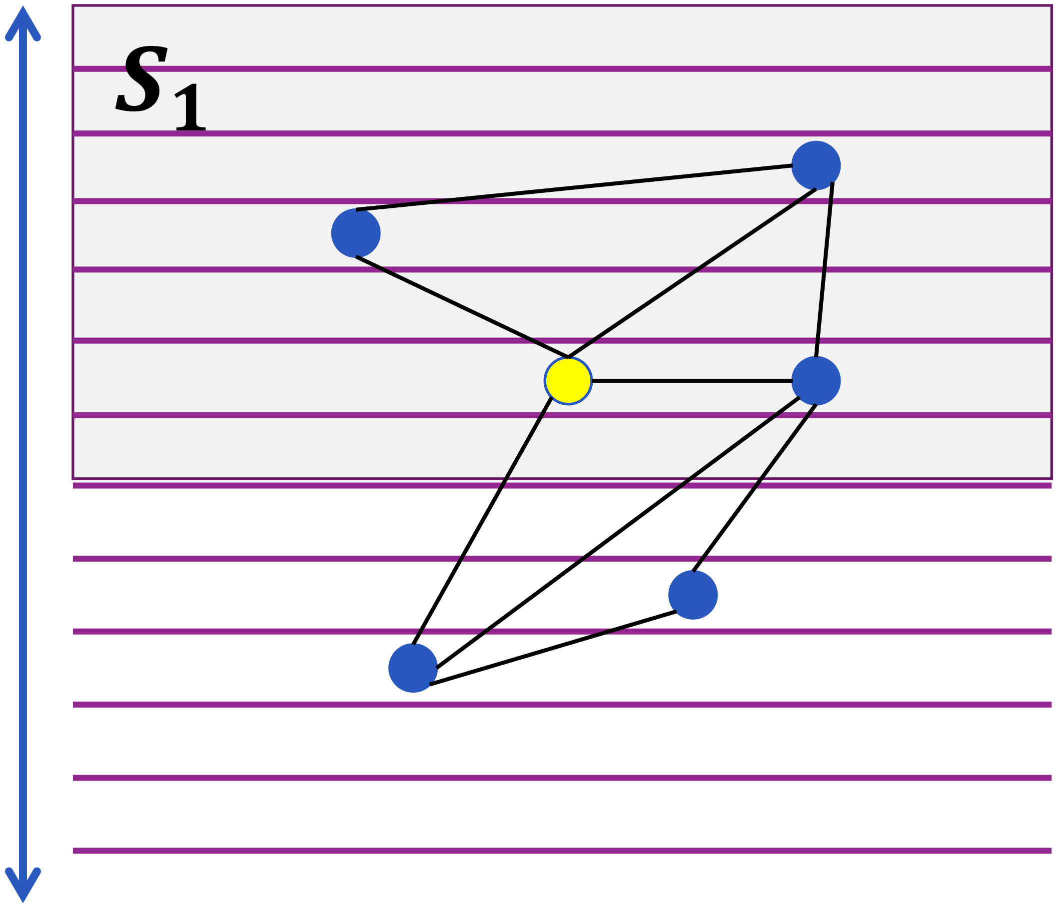
Proof of Our Approximation Factor: Lower Bound

At least $(1+\epsilon)^i - \frac{(1+\epsilon)^i}{2.5}$

edges must be pruned

Assume for Contradiction:

$$c(v) < \frac{(1+\epsilon)^i}{2.5}$$



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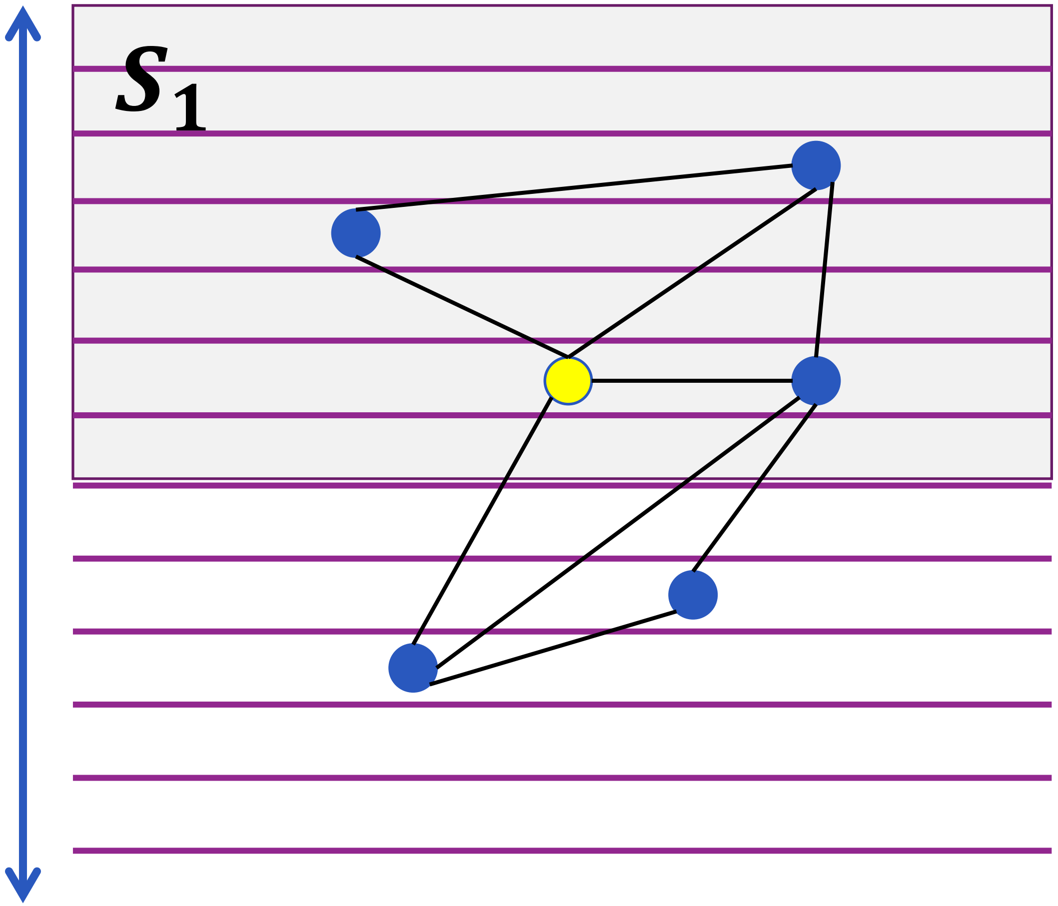

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Proof of Our Approximation Factor: Lower Bound

By Induction:

At least $\left(\frac{(1+\epsilon)^i}{2}\right)^j$ edges must be pruned

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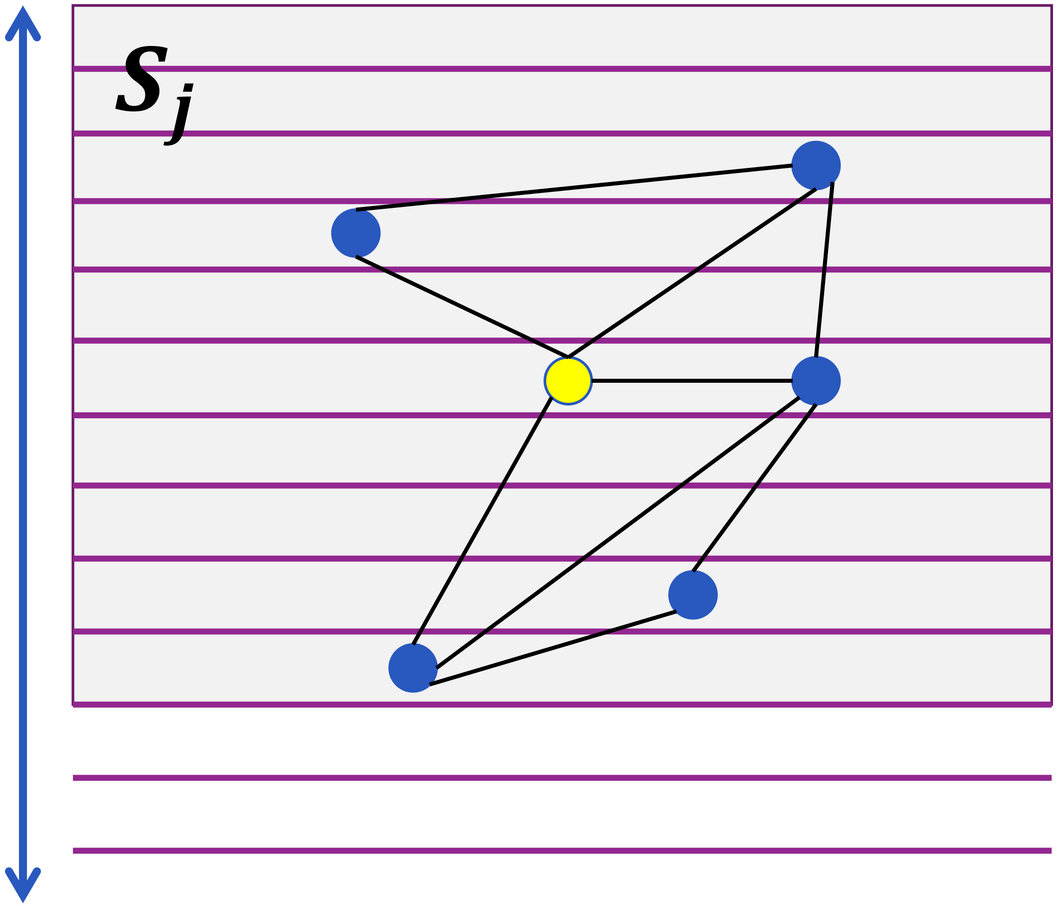

Pruning Procedure

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$$d_{S_j}(w) < \frac{(1+\epsilon)^i}{2.5}$$

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Proof of Our Approximation Factor: Lower Bound

By Induction:

At least $\left(\frac{(1+\epsilon)^i}{2}\right)^j$ edges must be pruned

At least $\frac{\left(\frac{(1+\epsilon)^i}{2}\right)^j}{\frac{(1+\epsilon)^i}{2.5}}$ **nodes** must be pruned

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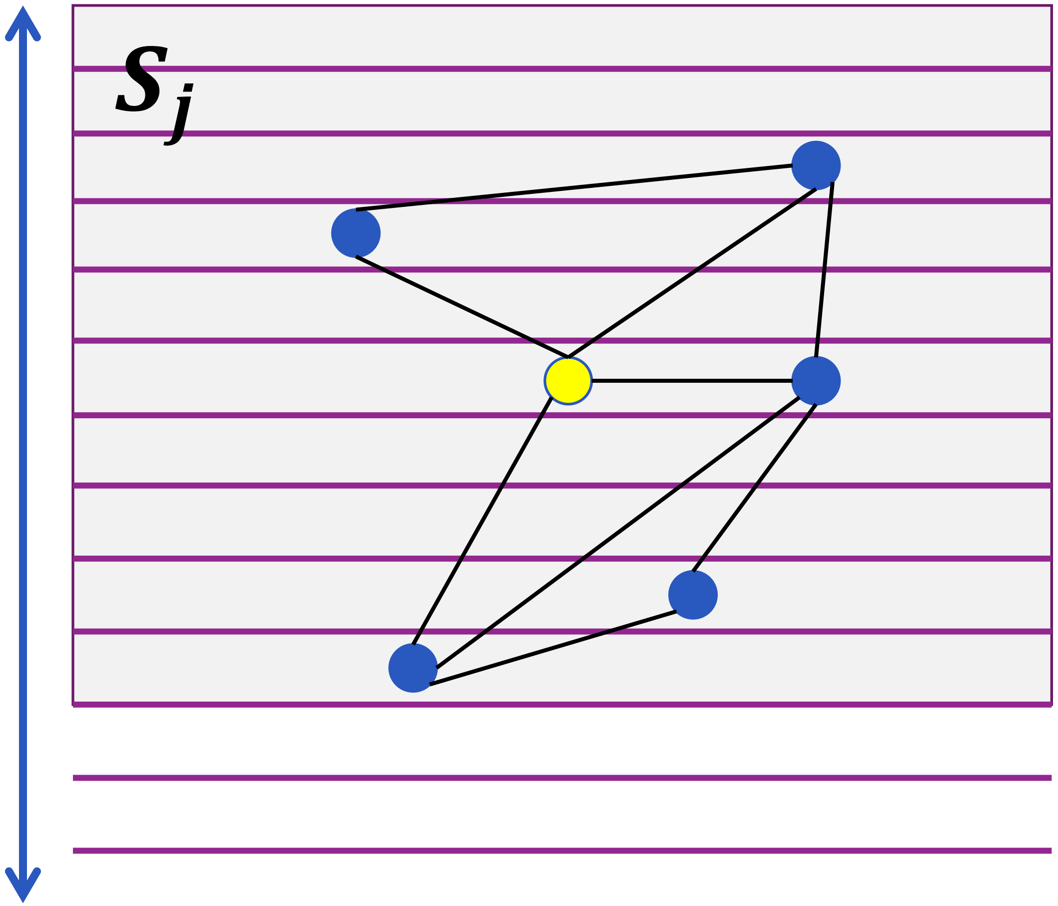

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At least $\left(\frac{(1+\epsilon)^i}{2}\right)^{j-1}$ **nodes** must be pruned

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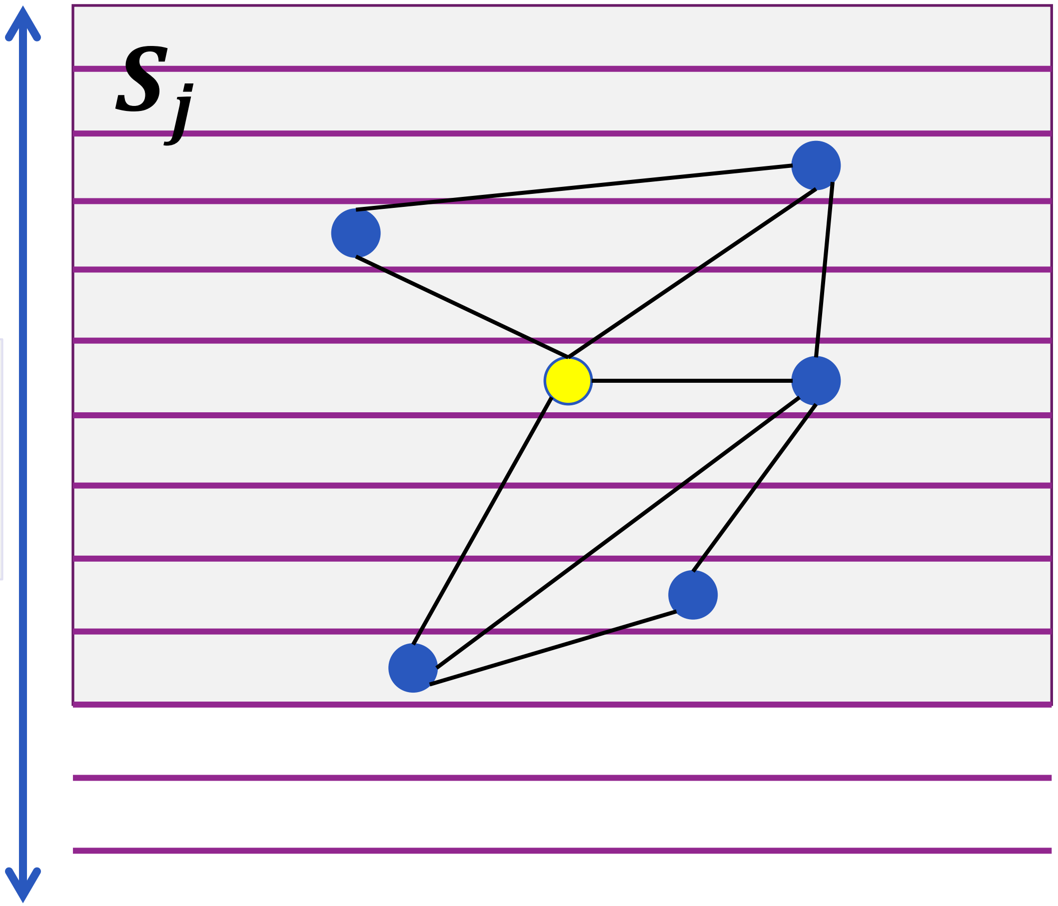

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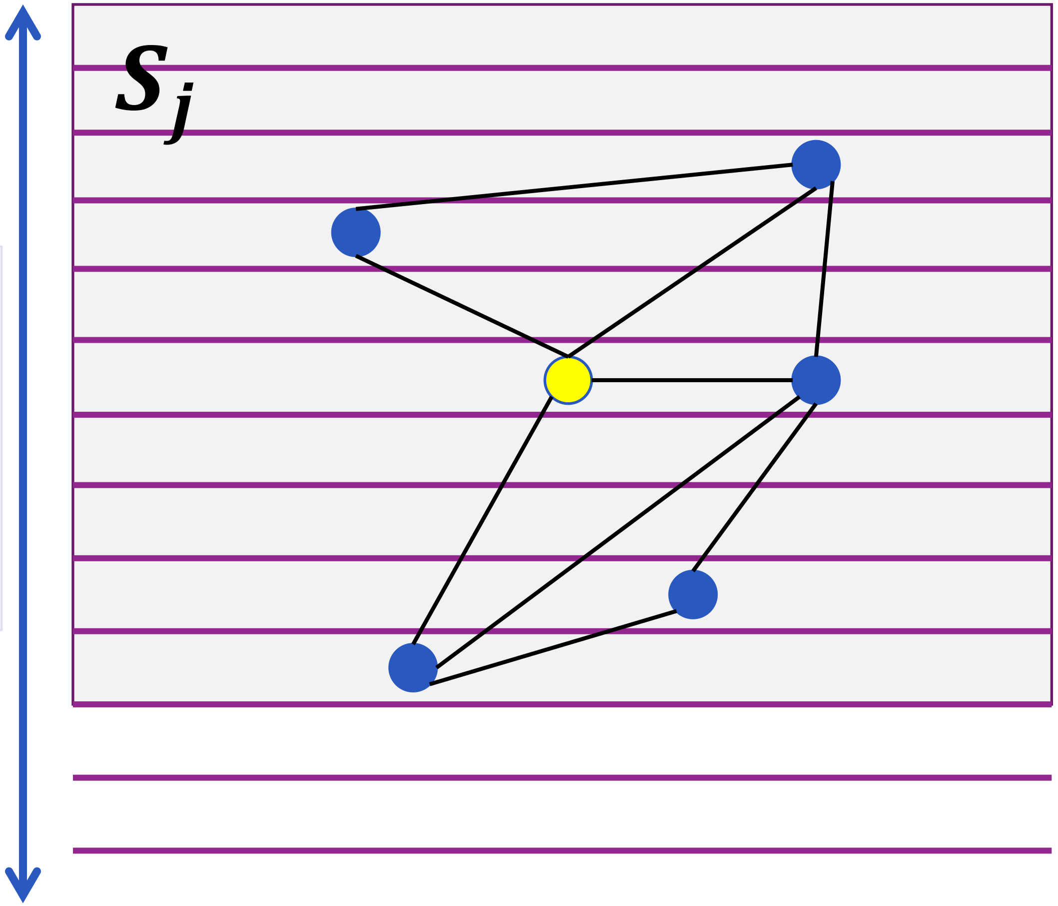
$$\left(\frac{(1+\epsilon)^i}{2}\right)^{j-1} \leq n$$

$$j \leq \log_{(1+\epsilon)^i/2}(n)$$

Assume for Contradiction:

$$c(v) < \frac{(1+\epsilon)^i}{2.5}$$



Pruning Procedure

Remove all w where

$$d_{S_j}(w) < \frac{(1+\epsilon)^i}{2.5}$$

nodes at or above level of v is: $\geq (1+\epsilon)^i$

Proof of Our Approximation Factor: Lower Bound

By Induction:

At least $\left(\frac{(1+\epsilon)^i}{2}\right)^j$

edges must be pruned

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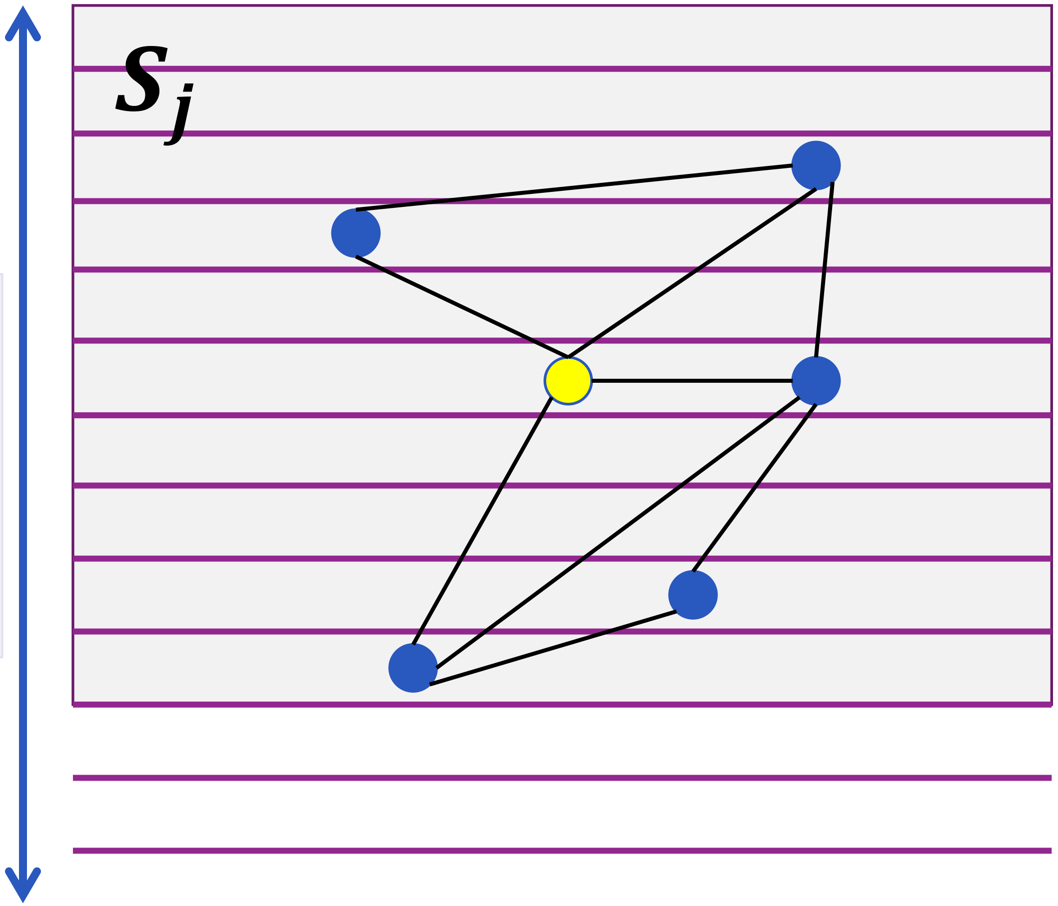

Pruning Procedure

Remove all w where

$$d_{S_j}(w) < \frac{(1+\epsilon)^i}{2.5}$$

nodes at or above level of v is: $\geq (1+\epsilon)^i$

Run out of vertices before first level of the group.

Proof of Our Approximation Factor: Lower Bound

By Induction:

At least $\left(\frac{(1+\epsilon)^i}{2}\right)^j$ edges must be pruned

$$\left(\frac{(1+\epsilon)^i}{2}\right)^{j-1} \leq n$$

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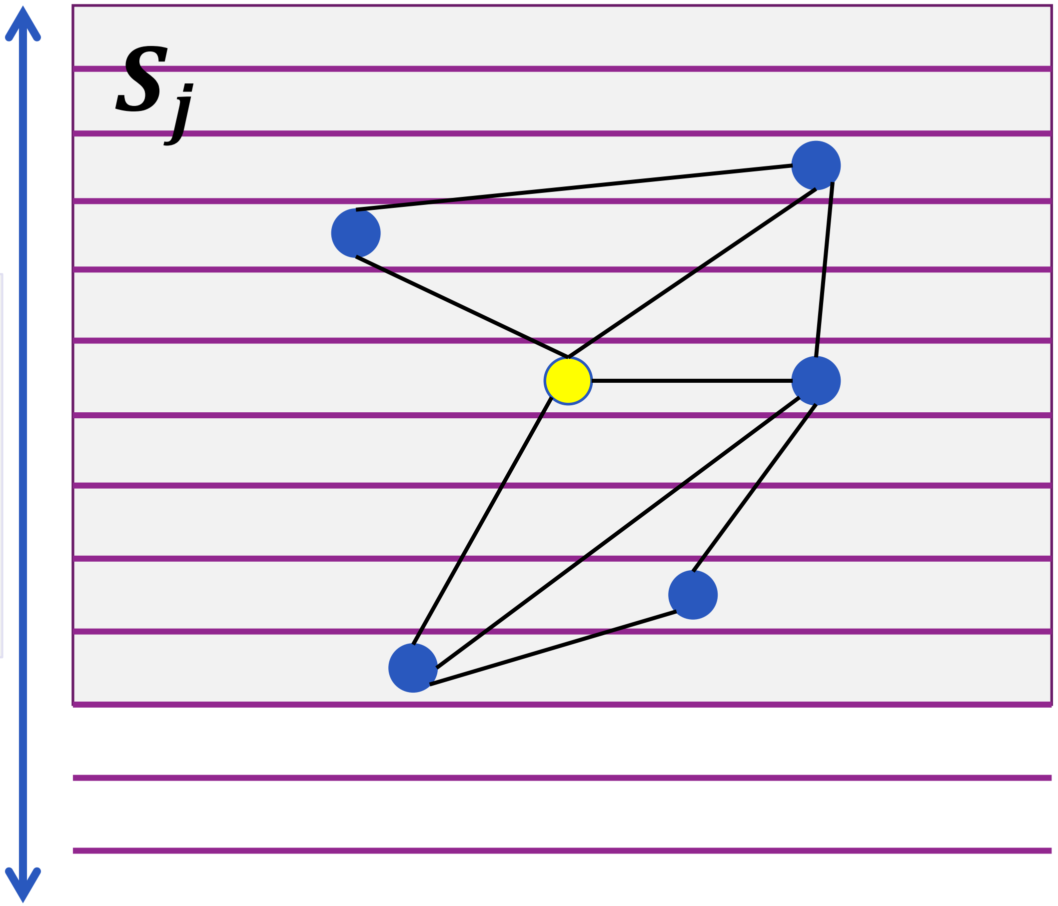

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